Math 530 Spring 2022, Lecture 5: Bridges and dominating sets

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Simple graphs (cont'd)

1.1. The longest path trick

Here is another proposition that guarantees the existence of cycles in a graph under certain circumstances. More importantly, its proof illustrates a useful tactic in dealing with graphs:

> **Proposition 1.1.1.** Let G be a simple graph with at least one vertex. Let $d > 1$ be an integer. Assume that each vertex of G has degree $\geq d$. Then, G has a cycle of length $\geq d + 1$.

Proof. Let $\mathbf{p} = (v_0, v_1, \ldots, v_m)$ be a **longest** path of G. (Why does G have a longest path? Let's see: Any path of G has length $\leq |V| - 1$, since its vertices have to be distinct. Moreover, G has at least one vertex and thus has at least one path. A finite nonempty set of integers has a largest element. Thus, G has a longest path.)

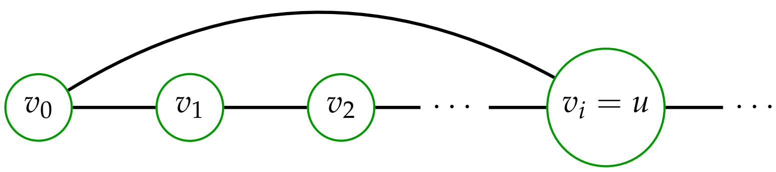
The vertex v_0 has degree $\geq d$ (by assumption), and thus has $\geq d$ neighbors (since the degree of a vertex is the number of its neighbors).

If all neighbors of v_0 belonged to the set $\{v_1, v_2, \ldots, v_{d-1}\}$ [1], then the number of neighbors of v_0 would be at most $d - 1$, which would contradict the previous sentence. Thus, there exists at least one neighbor u of v_0 that does **not** belong to this set $\{v_1, v_2, \ldots, v_{d-1}\}$. Consider this u. Then, $u \neq v_0$ (since a vertex cannot be its own neighbor).

Attaching the vertex u to the front of the path $\mathbf{p}$, we obtain a walk

$$\mathbf{p}' := (u, v_0, v_1, \ldots, v_m).$$

If we had $u \notin \{v_0, v_1, \ldots, v_m\}$, then this walk $\mathbf{p}'$ would be a path; but this would contradict the fact that $\mathbf{p}$ is a **longest** path of G. Thus, we must have $u \in \{v_0, v_1, \ldots, v_m\}$. In other words, $u = v_i$ for some $i \in \{0, 1, \ldots, m\}$. Consider this i. Since $u \neq v_0$ and $u \notin \{v_1, v_2, \ldots, v_{d-1}\}$, we thus have $i \geq d$. Here is a picture:



[1] If $d - 1 > m$, then this set should be understood to mean $\{v_1, v_2, \ldots, v_m\}$.

Now, consider the walk

$$\mathbf{c} := (u, v_0, v_1, \ldots, v_i).$$

This is a closed walk (since $u = v_i$) and has length $i + 1 \geq d + 1$ (since $i \geq d$). If we can show that $\mathbf{c}$ is a cycle, then we have thus found a cycle of length $\geq d + 1$, so we will be done.

It thus remains to prove that $\mathbf{c}$ is a cycle. Let us do this. We need to check that the vertices $u, v_0, v_1, \ldots, v_{i-1}$ are distinct, and that the length of $\mathbf{c}$ is ≥ 3. The latter claim is clear: The length of $\mathbf{c}$ is $i + 1 \geq d + 1 \geq 3$ (since $d > 1$ and $d \in \mathbb{Z}$). The former claim is not much harder: Since $u = v_i$, the vertices $u, v_0, v_1, \ldots, v_{i-1}$ are just the vertices $v_i, v_0, v_1, \ldots, v_{i-1}$, and thus are distinct because they are distinct vertices of the path $\mathbf{p}$. The proof of Proposition 1.1.1 is thus complete. □

1.2. Bridges

One question that will later prove crucial is: What happens to a graph if we remove a single edge from it? Let us first define a notation for this:

Definition 1.2.1. Let $G = (V, E)$ be a simple graph. Let e be an edge of G. Then, $G \setminus e$ will mean the graph obtained from G by removing this edge e. In other words,

$$G \setminus e := (V, \ E \setminus \{e\}).$$

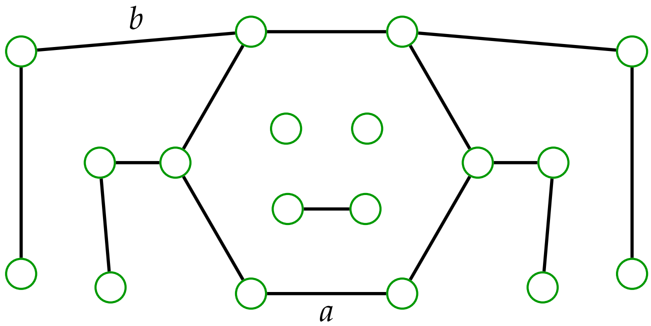
Some authors write $G - e$ for $G \setminus e$.

Theorem 1.2.2. Let G be a simple graph. Let e be an edge of G. Then:

(a) If e is an edge of some cycle of G, then the components of $G \setminus e$ are precisely the components of G. (Keep in mind that the components are sets of vertices. It is these sets that we are talking about here, not the induced subgraphs on these sets.)

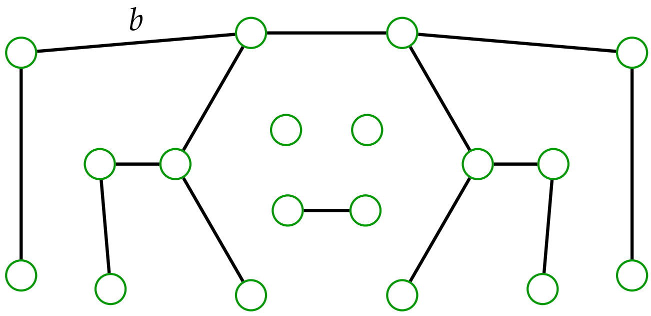
(b) If e appears in no cycle of G (in other words, there exists no cycle of G such that e is an edge of this cycle), then the graph $G \setminus e$ has one more component than G.

Example 1.2.3. Let G be the graph shown in the following picture:

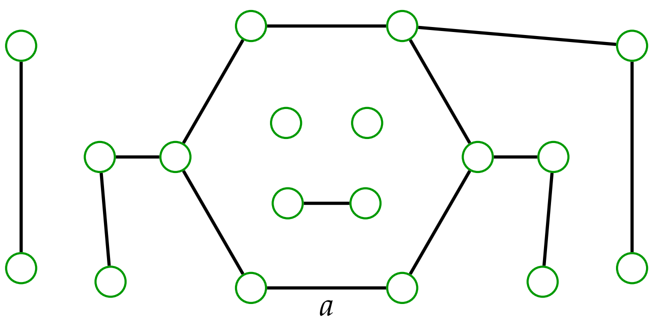


(1)

(where we have labeled the edges a and b for further reference). This graph has 4 components. The edge a is an edge of a cycle of G, whereas the edge b appears in no cycle of G. Thus, if we set $e = a$, then Theorem 1.2.2 **(a)** shows that the components of $G \setminus e$ are precisely the components of G. This graph $G \setminus e$ for $e = a$ looks as follows:



and visibly has 4 components. On the other hand, if we set $e = b$, then Theorem 1.2.2 **(b)** shows that the graph $G \setminus e$ has one more component than G. This graph $G \setminus e$ for $e = b$ looks as follows:



and visibly has 5 components.

Proof of Theorem 1.2.2. We will only sketch the proof. For details, see [21f6, §6.7].

Let u and v be the endpoints of e, so that $e = uv$. Note that (u, v) is a path of

G, and thus we have $u \simeq_G v$.

(a) Assume that e is an edge of some cycle of G. Then, if you remove e from this cycle, then you still have a path from u to v left (as the remaining edges of the cycle function as a detour), and this path is a path of $G \setminus e$. Thus, $u \simeq_{G\setminus e} v$.

Now, we must show that the components of $G \setminus e$ are precisely the components of G. This will clearly follow if we can show that the relation $\simeq_{G\setminus e}$ is precisely the relation $\simeq_G$ (because the components of a graph are the equivalence classes of its $\simeq$ relation). So let us prove the latter fact.

We must show that two vertices x and y of G satisfy $x \simeq_{G\setminus e} y$ if and only if they satisfy $x \simeq_G y$. The "only if" part is obvious (since a walk of $G \setminus e$ is always a walk of G). It thus remains to prove the "if" part. So we assume that x and y are two vertices of G satisfying $x \simeq_G y$, and we want to show that $x \simeq_{G\setminus e} y$.

From $x \simeq_G y$, we conclude that G has a path from x to y (by the second proposition in Lecture 4). If this path does not use[2] the edge e, then it is a path from x to y in $G \setminus e$, and thus we have $x \simeq_{G\setminus e} y$, which is what we wanted to prove. So we WLOG assume that this path does use the edge e. Thus, this path contains the endpoints u and v of this edge e. We WLOG assume that u appears before v on this path (otherwise, just swap u with v). Thus, this path looks as follows:

$$(x, \ldots, u, v, \ldots, y).$$

If we remove the edge $e = uv$, then this path breaks into two smaller paths

$$(x, \ldots, u) \qquad \text{and} \qquad (v, \ldots, y)$$

(since the edges of a path are distinct, so e appears only once in it). Both of these two smaller paths are paths of $G \setminus e$. Thus, $x \simeq_{G\setminus e} u$ and $v \simeq_{G\setminus e} y$. Now, recalling that $\simeq_{G\setminus e}$ is an equivalence relation, we combine these results to obtain

$$x \simeq_{G\setminus e} u \simeq_{G\setminus e} v \simeq_{G\setminus e} y.$$

Hence, $x \simeq_{G\setminus e} y$. This completes the proof of Theorem 1.2.2 **(a)**.

(b) Assume that e appears in no cycle of G. We must prove that the graph $G \setminus e$ has one more component than G. To do so, it suffices to show the following:

> *Claim 1:* The component of G that contains u and v (this component exists, since $u \simeq_G v$) breaks into two components of $G \setminus e$ when the edge e is removed.
>
> *Claim 2:* All other components of G remain components of $G \setminus e$.

Claim 2 is pretty clear: The components of G that don't contain u and v do not change at all when e is removed (since they contain neither endpoint of e). Thus, they remain components of $G \setminus e$. (Formalizing this is a nice exercise in formalization; see [21f6, §6.7].)

It remains to prove Claim 1. We introduce some notations:

[2] We say that a walk $\mathbf{w}$ **uses** an edge f if f is an edge of $\mathbf{w}$.

- Let C be the component of G that contains u and v.
- Let A be the component of $G \setminus e$ that contains u.
- Let B be the component of $G \setminus e$ that contains v.

Then, we must show that $A \cup B = C$ and $A \cap B = \varnothing$.

To see that $A \cap B = \varnothing$, we need to show that $u \simeq_{G \setminus e} v$ does **not** hold (since A and B are the equivalence classes of u and v with respect to the relation $\simeq_{G \setminus e}$). So let us do this. Assume the contrary. Thus, $u \simeq_{G \setminus e} v$. Hence, there exists a path from u to v in $G \setminus e$. Since $e = uv$, we can "close" this path by appending the vertex u to its end; the result is a cycle of the graph G that contains the edge e. But this contradicts our assumption that no cycle of G contains e. This contradiction shows that our assumption was wrong. Thus, we conclude that $u \simeq_{G \setminus e} v$ does **not** hold. Hence, as we said, $A \cap B = \varnothing$.

It remains to show that $A \cup B = C$. Since A and B are clearly subsets of C (because each walk of $G \setminus e$ is a walk of G, and thus each component of $G \setminus e$ is a subset of a component of G), we have $A \cup B \subseteq C$, and therefore we only need to show that $C \subseteq A \cup B$. In other words, we need to show that each $c \in C$ belongs to $A \cup B$.

Let us show this. Let $c \in C$ be a vertex. Then, $c \simeq_G u$ (since C is the component of G containing u). Therefore, G has a path $\mathbf{p}$ from c to u. Consider this path $\mathbf{p}$. Two cases are possible:

- *Case 1:* This path $\mathbf{p}$ does not use the edge e. In this case, $\mathbf{p}$ is a path of $G \setminus e$, and thus we obtain $c \simeq_{G \setminus e} u$. In other words, $c \in A$ (since A is the component of $G \setminus e$ containing u).
- *Case 2:* This path $\mathbf{p}$ does use the edge e. In this case, the edge e must be the last edge of $\mathbf{p}$ (since the path $\mathbf{p}$ would otherwise contain the vertex u twice[3]; but a path cannot contain a vertex twice), and the last two vertices of $\mathbf{p}$ must be v and u in this order. Thus, by removing the last vertex from $\mathbf{p}$, we obtain a path from c to v, and this latter path is a path of $G \setminus e$ (since it no longer contains u and therefore does not use e). This yields $c \simeq_{G \setminus e} v$. In other words, $c \in B$ (since B is the component of $G \setminus e$ containing v).

In either of these two cases, we have shown that c belongs to one of A and B. In other words, $c \in A \cup B$. This is precisely what we wanted to show. This completes the proof of Theorem 1.2.2 **(b)**. □

We introduce some fairly standard terminology:

[3]Indeed, the path $\mathbf{p}$ already ends in u. If it would contain e anywhere other than at the very end, then it would thus contain the vertex u twice (since u is an endpoint of e).

Definition 1.2.4. Let e be an edge of a simple graph G.

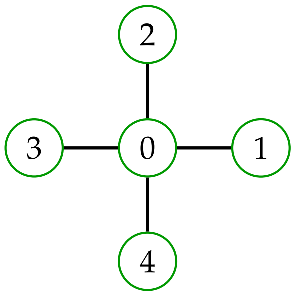
(a) We say that e is a **bridge** (of G) if e appears in no cycle of G.

(b) We say that e is a **cut-edge** (of G) if the graph $G \setminus e$ has more components than G.

Corollary 1.2.5. Let e be an edge of a simple graph G. Then, e is a bridge if and only if e is a cut-edge.

Proof. Follows from Theorem 1.2.2. □

We can also define "cut-vertices": A vertex v of a graph G is said to be a **cut-vertex** if the graph $G \setminus v$ (that is, the graph G with the vertex v removed[4]) has more components than G. Unfortunately, there doesn't seem to be an analogue of Corollary 1.2.5 for cut-vertices. Note also that removing a vertex (unlike removing an edge) can add more than one component to the graph (or it can also subtract 1 component if this vertex had degree 0). For example, removing the vertex 0 from the graph



results in an empty graph on the set $\{1, 2, 3, 4\}$, so the number of components has increased from 1 to 4.

1.3. Dominating sets

Here is another concept we can define for a graph:

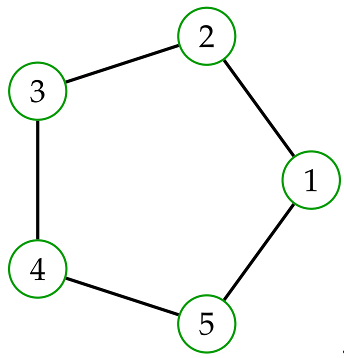
Definition 1.3.1. Let $G = (V, E)$ be a simple graph.

A subset U of V is said to be **dominating** (for G) if it has the following property: Each vertex $v \in V \setminus U$ has at least one neighbor in U.

A **dominating set** for G (or **dominating set** of G) will mean a subset of V that is dominating.

[4]When we remove a vertex, we must of course also remove all edges that contain this vertex.

Example 1.3.2. Consider the cycle graph

$$C_5 = (\{1,2,3,4,5\}, \ \{12, \ 23, \ 34, \ 45, \ 51\}) =$$ 

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The set $\{1,3\}$ is a dominating set for C_5, since all three vertices $2,4,5$ that don't belong to $\{1,3\}$ have neighbors in $\{1,3\}$. The set $\{1,5\}$ is not a dominating set for C_5, since the vertex 3 has no neighbor in $\{1,5\}$. There is no dominating set for C_5 that has size 0 or 1, but there are several of size 2, and every subset of size ≥ 3 is dominating.

Here are some more examples:

- If $G = (V, E)$ is a simple graph, then the whole vertex set V is always dominating, whereas the empty set $\varnothing$ is dominating only when $V = \varnothing$.
- If $G = (V, E)$ is a complete graph, then any nonempty subset of V is dominating.
- If $G = (V, E)$ is an empty graph, then only V is dominating.

Clearly, the "denser" a graph is (i.e., the more edges it has), the "easier" it is for a set to be dominating. Often, a graph is given, and one is interested in finding a dominating set of the smallest possible size[5]. As the case of an empty graph reveals, sometimes the only choice is the whole vertex set. However, in many cases, we can do better. Namely, we need to require that the graph has no isolated vertices:

Definition 1.3.3. Let G be a simple graph. A vertex v of G is said to be **isolated** if it has no neighbors (i.e., if $\deg v = 0$).

An isolated vertex has to belong to every dominating set (since otherwise, it would need a neighbor in that set, but it has no neighbors). Thus, isolated vertices do not contribute much to the study of dominating sets, other than inflating their size. Therefore, when we look for dominating sets, we can restrict ourselves to graphs with no isolated vertices. There, we have the following result:

[5]Supposedly, this has applications in mobile networking: For example, you might want to choose a set of routers in a given network so that each node is either a router or directly connected (i.e., adjacent) to one.

Proposition 1.3.4. Let $G = (V, E)$ be a simple graph that has no isolated vertices. Then:

(a) There exists a dominating subset of V that has size $\leq |V| / 2$.

(b) There exist two disjoint dominating subsets A and B of V such that $A \cup B = V$.

One proof of this proposition will be given in homework set #2 exercise 4. Another appears in [17s, §3.6].

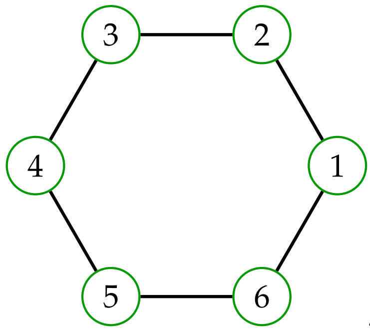
Next, we state a rather surprising recent result about the number of dominating sets of a graph (which, I believe, appeared as a Putnam problem a couple decades ago):

Theorem 1.3.5 (Brouwer's dominating set theorem)**.** Let G be a simple graph. Then, the number of dominating sets of G is odd.

Three proofs of this theorem are given in Brouwer's note [Brouwe09]. Let me show the one I like the most. We first need a notation:

Definition 1.3.6. Let $G = (V, E)$ be a simple graph. A **detached pair** will mean a pair (A, B) of two disjoint subsets A and B of V such that there exists no edge $ab \in E$ with $a \in A$ and $b \in B$.

Example 1.3.7. Consider the cycle graph

$C_6 = (\{1,2,3,4,5,6\}, \{12, 23, 34, 45, 56, 61\}) =$  .

Then, $(\{1,2\}, \{4,5\})$ is a detached pair, whereas $(\{1,2\}, \{3,4\})$ is not (since 23 is an edge). Of course, there are many other detached pairs; in particular, any pair of the form $(\varnothing, B)$ or $(A, \varnothing)$ is detached.

Let me stress that the word "pair" always means "ordered pair" unless I say otherwise. So, if (A, B) is a detached pair, then (B, A) is a different detached pair, unless $A = B = \varnothing$.

Here is an attempt at a proof of Theorem 1.3.5. It is a nice example of how to apply known results to new graphs to obtain new results. The only problem is, it shows a result that is a bit at odds with the claim of the theorem...

Proof of Brouwer's dominating set theorem, attempt 1. Write the graph G as (V, E).

Recall that $\mathcal{P}(V)$ denotes the set of all subsets of V.

Construct a new graph H with the vertex set $\mathcal{P}(V)$ as follows: Two subsets A and B of V are adjacent as vertices of H if and only if (A, B) is a detached pair. (Note that if the original graph G has n vertices, then this graph H has 2^n vertices. It is huge!)

I claim that the vertices of H that have odd degree are precisely the subsets of V that are dominating. In other words:

> *Claim 1:* Let A be a subset of V. Then, the vertex A of H has odd degree if and only if A is a dominating set of G.

[*Proof of Claim 1:* We let $N(A)$ denote the set of all vertices of G that have a neighbor in A. (This may or may not be disjoint from A.)

The neighbors of A (as a vertex in H) are precisely the subsets B of V such that (A, B) is a detached pair (by the definition of H). In other words, they are the subsets B of V that are disjoint from A and also have no neighbors in A (by the definition of a "detached pair"). In other words, they are the subsets B of V that are disjoint from A and also disjoint from $N(A)$. In other words, they are the subsets of the set $V \setminus (A \cup N(A))$. Hence, the number of such subsets B is $2^{|V \setminus (A \cup N(A))|}$.

The degree of A (as a vertex of H) is the number of neighbors of A in H. Thus, this degree is $2^{|V \setminus (A \cup N(A))|}$ (because we have just shown that the number of neighbors of A is $2^{|V \setminus (A \cup N(A))|}$). But 2^k is odd if and only if $k = 0$. Thus, we conclude that the degree of A (as a vertex of H) is odd if and only if $|V \setminus (A \cup N(A))| = 0$. The condition $|V \setminus (A \cup N(A))| = 0$ can be rewritten as follows:

$$
\begin{aligned}
& (|V \setminus (A \cup N(A))| = 0) \\
\iff & (V \setminus (A \cup N(A)) = \varnothing) \\
\iff & (V \subseteq A \cup N(A)) \\
\iff & (V \setminus A \subseteq N(A)) \\
\iff & (\text{each vertex } v \in V \setminus A \text{ belongs to } N(A)) \\
\iff & (\text{each vertex } v \in V \setminus A \text{ has a neighbor in } A) \\
\iff & (A \text{ is dominating}) \qquad (\text{by the definition of "dominating"}).
\end{aligned}
$$

Thus, what we have just shown is that the degree of A (as a vertex of H) is odd if and only if A is dominating. This proves Claim 1.]

Claim 1 shows that the vertices of H that have odd degree are precisely the dominating sets of G. But the handshake lemma (the first corollary in Lecture 2) tells us that any simple graph has an even number of vertices of odd degree. Applying this to H, we conclude that there is an even number of dominating sets of G.

Huh? We want to show that there is an **odd** number of dominating sets of G, not an even number! Why did we just get the opposite result?

Homework: Find the mistake in our above reasoning! (I will also reveal it next time.) □

References

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[21f6] Darij Grinberg, *Math 235 Fall 2021, Worksheet 6: Graphs and some of their uses*, 6 April 2022.
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[Brouwe09] Andries E. Brouwer, *The number of dominating sets of a finite graph is odd*, `http://www.win.tue.nl/~aeb/preprints/domin2.pdf` .