Math 530 Spring 2022, Lecture 4: Cycles, connectivity

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Simple graphs (cont'd)

1.1. Walks and paths (cont'd)

Last time, we defined the concepts of walks and paths in a simple graph. Here are their intuitive meanings:

- A **walk** of a graph is a way of walking from one vertex to another (or to the same vertex) by following a sequence of edges.
- A **path** is a walk whose vertices are distinct (i.e., each vertex appears at most once in the walk).

1.1.1. The equivalence relation "path-connected"

We can use the concepts of walks and paths to define a certain equivalence relation on the vertex set $\operatorname{V}(G)$ of any graph G:

Definition 1.1.1. Let G be a simple graph. We define a binary relation $\simeq_G$ on the set $\operatorname{V}(G)$ as follows: For two vertices u and v of G, we shall have $u \simeq_G v$ if and only if there exists a walk from u to v in G.

This binary relation $\simeq_G$ is called "**path-connectedness**" or just "**connectedness**". When two vertices u and v satisfy $u \simeq_G v$, we say that "u and v are **path-connected**".

Proposition 1.1.2. Let G be a simple graph. Then, the relation $\simeq_G$ is an equivalence relation.

Proof. We need to show that $\simeq_G$ is symmetric, reflexive and transitive.

- **Symmetry:** If $u \simeq_G v$, then $v \simeq_G u$, because we can take a walk from u to v and reverse it.
- **Reflexivity:** We always have $u \simeq_G u$, since the trivial walk (u) is a walk from u to u.
- **Transitivity:** If $u \simeq_G v$ and $v \simeq_G w$, then $u \simeq_G w$, because (as we showed in Lecture 3) we can take a walk $\mathbf{a}$ from u to v and a walk $\mathbf{b}$ from v to w and combine them to form the walk $\mathbf{a} * \mathbf{b}$ defined in Lecture 3.

$\square$

Proposition 1.1.3. Let G be a simple graph. Let u and v be two vertices of G. Then, $u \simeq_G v$ if and only if there exists a path from u to v.

Proof. $\Longleftarrow$: Clear, since any path is a walk.

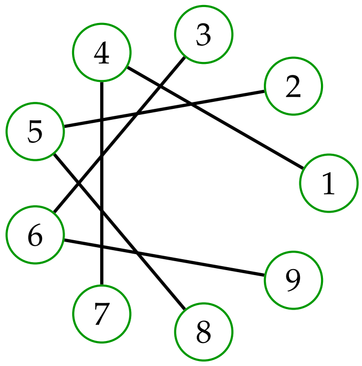
$\Longrightarrow$: This is just saying that if there is a walk from u to v, then there is a path from u to v. But this follows from the last corollary of Lecture 3. $\square$

Definition 1.1.4. Let G be a simple graph. The equivalence classes of the equivalence relation $\simeq_G$ are called the **connected components** (or, for short, **components**) of G.

Definition 1.1.5. Let G be a simple graph. We say that G is **connected** if G has exactly one component.

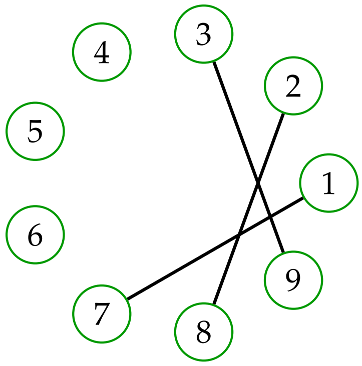
Example 1.1.6. Let G be the graph with vertex set $\{1, 2, \ldots, 9\}$ and such that two vertices i and j are adjacent if and only if $|i - j| = 3$. What are the components of G ?

The graph G looks like this:



This looks like a jumbled mess, so you might think that all vertices are mutually path-connected. But this is not the case, because edges that cross in a drawing do not necessarily have endpoints in common. Walks can only move from one edge to another at a common endpoint. Thus, there are much fewer walks than the picture might suggest. We have $1 \simeq_G 4 \simeq_G 7$ and $2 \simeq_G 5 \simeq_G 8$ and $3 \simeq_G 6 \simeq_G 9$, but there are no further $\simeq_G$-relations. In fact, two vertices of G are adjacent only if they are congruent modulo 3 (as numbers), and therefore you cannot move from one modulo-3 congruence class to another by walking along edges of G. So the components of G are $\{1, 4, 7\}$ and $\{2, 5, 8\}$ and $\{3, 6, 9\}$. The graph G is not connected.

Example 1.1.7. Let G be the graph with vertex set $\{1, 2, \ldots, 9\}$ and such that two vertices i and j are adjacent if and only if $|i - j| = 6$. This graph looks like this:



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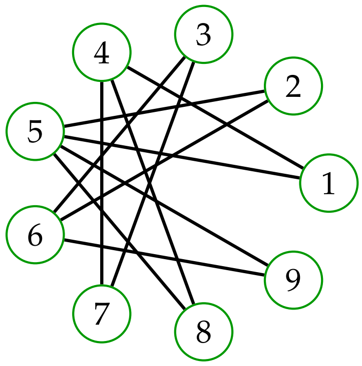
What are the components of G ? They are $\{1,7\}$ and $\{2,8\}$ and $\{3,9\}$ and $\{4\}$ and $\{5\}$ and $\{6\}$. Note that three of these six components are singleton sets. The graph G is not connected.

Example 1.1.8. Let G be the graph with vertex set $\{1,2,\ldots,9\}$ and such that two vertices i and j are adjacent if and only if $|i-j|=3$ or $|i-j|=4$. This graph looks like this:



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We can take a long walk through G:

$$(1,4,7,3,6,9,5,2,5,8).$$

This walk traverses every vertex of G; thus, any two vertices of G are path-connected. Hence, G has only one component, namely $\{1,2,\ldots,9\}$. Thus, G is connected.

Example 1.1.9. The complete graph on a nonempty set is connected. The complete graph on the empty set is not connected, since it has 0 (not 1) components.

Example 1.1.10. The empty graph on a finite set V has $|V|$ many components (those are the singleton sets $\{v\}$ for $v\in V$). Thus, it is connected if and only if $|V|=1$.

The following is not hard to see:

Proposition 1.1.11. Let G be a simple graph. Let C be a component of G. Then, the induced subgraph of G on the set C is connected.

Proof. Let $G[C]$ be this induced subgraph. We need to show that $G[C]$ is connected. In other words, we need to show that $G[C]$ has exactly 1 component.

Clearly, $G[C]$ has at least one vertex (since C is a component, i.e., an equivalence class of $\simeq_G$, but equivalence classes are always nonempty), thus has at least 1 component. So we only need to show that $G[C]$ has no more than 1 component. In other words, we need to show that any two vertices of $G[C]$ are path-connected in $G[C]$.

So let u and v be two vertices of $G[C]$. Then, $u, v \in C$, and therefore $u \simeq_G v$ (since C is a component of G). In other words, there exists a walk $\mathbf{w} = (w_0, w_1, \ldots, w_k)$ from u to v in G. We shall now prove that this walk $\mathbf{w}$ is actually a walk of $G[C]$. In other words, we shall prove that all vertices of $\mathbf{w}$ belong to C.

But this is easy: If w_i is a vertex of $\mathbf{w}$, then $(w_0, w_1, \ldots, w_i)$ is a walk from u to w_i in G, and therefore we have $u \simeq_G w_i$, so that w_i belongs to the same component of G as u; but that component is C. Thus, we have shown that each vertex w_i of $\mathbf{w}$ belongs to C. Therefore, $\mathbf{w}$ is a walk of the graph $G[C]$. Consequently, it shows that $u \simeq_{G[C]} v$.

We have now proved that $u \simeq_{G[C]} v$ for any two vertices u and v of $G[C]$. Hence, the relation $\simeq_{G[C]}$ has no more than 1 equivalence class. In other words, the graph $G[C]$ has no more than 1 component. This completes our proof. □

In the following proposition, we are using the notation $G[S]$ for the induced subgraph of a simple graph G on a subset S of its vertex set.

Proposition 1.1.12. Let G be a simple graph. Let $C_1, C_2, \ldots, C_k$ be all components of G (listed without repetition).

Thus, G is isomorphic to the disjoint union $G[C_1] \sqcup G[C_2] \sqcup \cdots \sqcup G[C_k]$.

Proof. Consider the bijection from $\operatorname{V}(G[C_1] \sqcup G[C_2] \sqcup \cdots \sqcup G[C_k])$ to $\operatorname{V}(G)$ that sends each vertex (i, v) of $G[C_1] \sqcup G[C_2] \sqcup \cdots \sqcup G[C_k]$ to the vertex v of G. We claim that this bijection is a graph isomorphism. In order to prove this, we need to check that there are no edges of G that join vertices in different components. But this is easy: If two vertices in different components of G were adjacent, then they would be path-connected, and thus would actually belong to the same component. □

The upshot of these results is that every simple graph can be decomposed into a disjoint union of its components (or, more precisely, of the induced subgraphs on its components). Each of these components is a connected graph. Moreover, this is easily seen to be the only way to decompose the graph into a disjoint union of connected graphs.

1.2. Closed walks and cycles

Here are two further kinds of walks:

Definition 1.2.1. Let G be a simple graph.

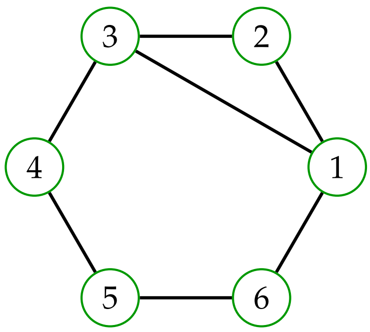
(a) A **closed walk** of G means a walk whose first vertex is identical with its last vertex. In other words, it means a walk $(w_0, w_1, \ldots, w_k)$ with $w_0 = w_k$. Sometimes, closed walks are also known as **circuits** (but many authors use this latter word for something slightly different).

(b) A **cycle** of G means a closed walk $(w_0, w_1, \ldots, w_k)$ such that $k \geq 3$ and such that the vertices $w_0, w_1, \ldots, w_{k-1}$ are distinct.

Example 1.2.2. Let G be the simple graph

$$(\{1,2,3,4,5,6\}, \ \{12, \ 23, \ 34, \ 45, \ 56, \ 61, \ 13\}).$$

This graph looks as follows (we have already seen it in Lecture 3):



Then:

- The sequence $(1,3,2,1,6,5,6,1)$ is a closed walk of G. But it is very much not a cycle.
- The sequences $(1,2,3,1)$ and $(1,3,4,5,6,1)$ and $(1,2,3,4,5,6,1)$ are cycles of G. You can get further cycles by rotating these sequences (in a proper sense of this word – e.g., rotating $(1,2,3,1)$ gives $(2,3,1,2)$ and $(3,1,2,3)$) and by reversing them. Every cycle of G can be obtained in this way.
- The sequences (1) and $(1,2,1)$ are closed walks, but not cycles of G (since they fail the $k \geq 3$ condition).
- The sequence $(1,2,3)$ is a walk, but not a closed walk, since $1 \neq 3$.

Authors have different opinions about whether $(1,2,3,1)$ and $(1,3,2,1)$ count as different cycles. Fortunately, this matters only if you want to count cycles, but not for the existence or non-existence of cycles.

We have now defined paths (in an arbitrary graph) and also path graphs P_n; we have also defined cycles (in an arbitrary graph) and also cycle graphs C_n. Besides their similar names, are they related? The answer is "yes":

Proposition 1.2.3. Let G be a simple graph.

(a) If $(p_0, p_1, \ldots, p_k)$ is a path of G, then there is a subgraph of G isomorphic to the path graph P_{k+1}, namely the subgraph $(\{p_0, p_1, \ldots, p_k\}, \{p_i p_{i+1} \mid 0 \leq i < k\})$. (If this subgraph is actually an induced subgraph of G, then the path $(p_0, p_1, \ldots, p_k)$ is called an "induced path".)

Conversely, any subgraph of G isomorphic to P_{k+1} gives a path of G.

(b) Now, assume that $k \geq 3$. If $(c_0, c_1, \ldots, c_k)$ is a cycle of G, then there is a subgraph of G isomorphic to the cycle graph C_k, namely the subgraph $(\{c_0, c_1, \ldots, c_k\}, \{c_i c_{i+1} \mid 0 \leq i < k\})$. (If this subgraph is actually an induced subgraph of G, then the cycle $(c_0, c_1, \ldots, c_k)$ is called an "induced cycle".)

Conversely, any subgraph of G isomorphic to C_k gives a cycle of G.

Proof. Straightforward. □

Certain graphs contain cycles; other graphs don't. For instance, the complete graph K_n contains a lot of cycles (when $n \geq 3$), whereas the path graph P_n contains none. Let us try to find some criteria for when a graph can and when it cannot have cycles[1]:

Proposition 1.2.4. Let G be a simple graph. Let $\mathbf{w}$ be a walk of G such that no two adjacent edges of $\mathbf{w}$ are identical. (By "adjacent edges", we mean edges of the form $w_{i-1}w_i$ and $w_i w_{i+1}$, where w_{i-1}, w_i, w_{i+1} are three consecutive vertices of $\mathbf{w}$.)

Then, $\mathbf{w}$ either is a path or contains a cycle (i.e., there exists a cycle of G whose edges are edges of $\mathbf{w}$).

Example 1.2.5. Let G be as in Example 1.2.2. Then, $(2,1,3,2,1,6)$ is a walk $\mathbf{w}$ of G such that no two adjacent edges of $\mathbf{w}$ are identical (even though the edge 21 appears twice in this walk). On the other hand, $(2,1,3,1,6)$ is not such a walk (since its two adjacent edges 13 and 31 are identical).

[1] Mantel's theorem already gives such a criterion for cycles of length 3 (because a cycle of length 3 is the same as a triangle).

Proof of Proposition 1.2.4. We assume that $\mathbf{w}$ is not a path. We must then show that $\mathbf{w}$ contains a cycle.

Write $\mathbf{w}$ as $\mathbf{w} = (w_0, w_1, \ldots, w_k)$. Since $\mathbf{w}$ is not a path, two of the vertices $w_0, w_1, \ldots, w_k$ must be equal. In other words, there exists a pair (i, j) of integers i and j with $i < j$ and $w_i = w_j$. Among all such pairs, we pick one with **minimum** difference $j - i$. We shall show that the walk $(w_i, w_{i+1}, \ldots, w_j)$ is a cycle.

First, this walk is clearly a closed walk (since $w_i = w_j$). It thus remains to show that $j - i \geq 3$ and that the vertices $w_i, w_{i+1}, \ldots, w_{j-1}$ are distinct. The distinctness of $w_i, w_{i+1}, \ldots, w_{j-1}$ follows from the minimality of $j - i$. To show that $j - i \geq 3$, we assume the contrary. Thus, $j - i$ is either 1 or 2 (since $i < j$). But $j - i$ cannot be 1, since the endpoints of an edge cannot be equal (since our graph is a simple graph). So $j - i$ must be 2. Thus, $w_i = w_{i+2}$. Therefore, the two edges $w_i w_{i+1}$ and $w_{i+1} w_{i+2}$ are identical. But this contradicts the fact that no two adjacent edges of $\mathbf{w}$ are identical. Contradiction, qed. □

Corollary 1.2.6. Let G be a simple graph. Assume that G has a closed walk $\mathbf{w}$ of length > 0 such that no two adjacent edges of $\mathbf{w}$ are identical. Then, G has a cycle.

Proof. This follows from Proposition 1.2.4, since $\mathbf{w}$ is not a path. □

Theorem 1.2.7. Let G be a simple graph. Let u and v be two vertices in G. Assume that there are two distinct paths from u to v. Then, G has a cycle.

Proof. More generally, we shall prove this theorem with the word "path" replaced by "backtrack-free walk", where a "**backtrack-free walk**" means a walk $\mathbf{w}$ such that no two adjacent edges of $\mathbf{w}$ are identical. This is a generalization of the theorem, since every path is a backtrack-free walk (why?).

So we claim the following:

> *Claim 1:* Let $\mathbf{p}$ and $\mathbf{q}$ be two distinct backtrack-free walks that start at the same vertex and end at the same vertex. Then, G has a cycle.

We shall prove Claim 1 by induction on the length of $\mathbf{p}$. So we fix an integer N, and we assume that Claim 1 is proved in the case when the length of $\mathbf{p}$ is $N - 1$. We must now show that it is also true when the length of $\mathbf{p}$ is N.

So let $\mathbf{p} = (p_0, p_1, \ldots, p_a)$ and $\mathbf{q} = (q_0, q_1, \ldots, q_b)$ be two distinct backtrack-free walks that start at the same vertex and end at the same vertex and satisfy $a = N$. We must find a cycle.

The walks $\mathbf{p}$ and $\mathbf{q}$ are distinct but start at the same vertex, so they cannot both be trivial[2]. If one of them is trivial, then the other is a closed walk (because a trivial walk is a closed walk), and then our goal follows from Corollary 1.2.6

[2]We say that a walk is **trivial** if it has length 0.

in this case (because we have a nontrivial closed backtrack-free walk). Hence, from now on, we WLOG assume that neither of the two walks $\mathbf{p}$ and $\mathbf{q}$ is trivial. Thus, each of these two walks has a last edge. The last edge of $\mathbf{p}$ is $p_{a-1}p_a$, whereas the last edge of $\mathbf{q}$ is $q_{b-1}q_b$.

Two cases are possible:

Case 1: We have $p_{a-1}p_a = q_{b-1}q_b$.

Case 2: We have $p_{a-1}p_a \neq q_{b-1}q_b$.

Let us consider Case 1 first. In this case, the last edges $p_{a-1}p_a$ and $q_{b-1}q_b$ of the two walks $\mathbf{p}$ and $\mathbf{q}$ are identical, so the second-to-last vertices of these two walks must also be identical. Thus, if we remove these last edges from both walks, then we obtain two shorter backtrack-free walks $(p_0, p_1, \ldots, p_{a-1})$ and $(q_0, q_1, \ldots, q_{b-1})$ that again start at the same vertex and end at the same vertex, but the length of the first of them is $a - 1 = N - 1$. Hence, by the induction hypothesis, we can apply Claim 1 to these two shorter walks (instead of $\mathbf{p}$ and $\mathbf{q}$), and we conclude that G has a cycle. So we are done in Case 1.

Let us now consider Case 2. In this case, we combine the two walks $\mathbf{p}$ and $\mathbf{q}$ (more precisely, $\mathbf{p}$ and the reversal of $\mathbf{q}$) to obtain the closed walk

$$(p_0, p_1, \ldots, p_{a-1}, p_a = q_b, q_{b-1}, \ldots, q_0).$$

This closed walk is backtrack-free (since $(p_0, p_1, \ldots, p_a)$ and $(q_0, q_1, \ldots, q_b)$ are backtrack-free, and since $p_{a-1}p_a \neq q_{b-1}q_b$) and has length > 0 (since it contains at least the edge $p_{a-1}p_a$). Hence, Corollary 1.2.6 entails that G has a cycle.

We have thus found a cycle in both Cases 1 and 2. This completes the induction step. Thus, we have proved Claim 1. As we said, Theorem 1.2.7 follows from it. $\square$