Math 530 Spring 2022, Lecture 3: Walks and paths

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Simple graphs (cont'd)

1.1. Some families of graphs

We will now define some particularly significant families of graphs.

1.1.1. Complete and empty graphs

The simplest families of graphs are the complete graphs and the empty graphs:

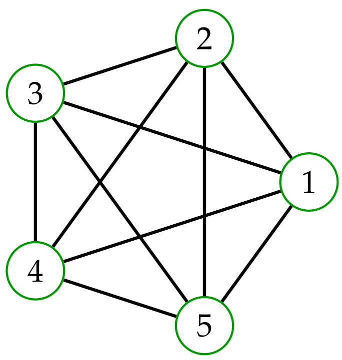
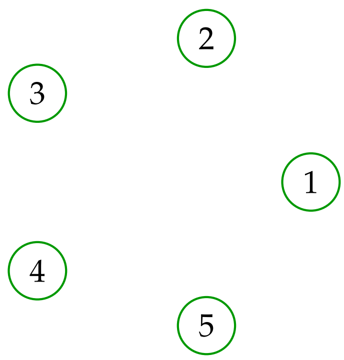
Definition 1.1.1. Let V be a finite set.

(a) The **complete graph** on V means the simple graph $(V, \mathcal{P}_2(V))$. It is the simple graph with vertex set V in which every two distinct vertices are adjacent.

If $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, then the complete graph on V is denoted K_n.

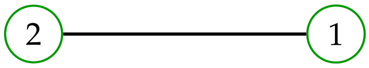
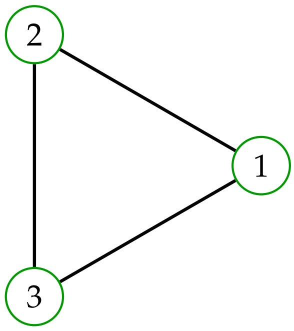
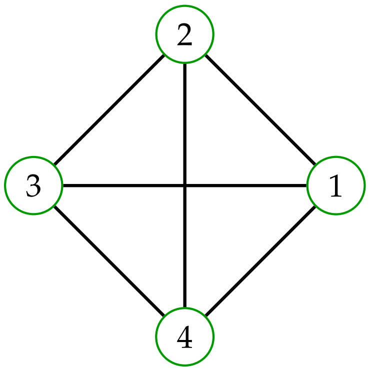
(b) The **empty graph** on V means the simple graph $(V, \varnothing)$. It is the simple graph with vertex set V and no edges.

The following pictures show the complete graph and the empty graph on the set $\{1, 2, 3, 4, 5\}$:

complete graph	empty graph
	

The complete one is called K_5.

Here are the complete graphs K_0, K_1, K_2, K_3, K_4:

K_0	
K_1	
K_2	
K_3	
K_4	

Note that a simple graph G is isomorphic to the complete graph K_n if and only if it has n vertices and is a complete graph (i.e., every two distinct vertices are adjacent).

Question: Given two finite sets V and W, what are the isomorphisms from the complete graph on V to the complete graph on W ?

Answer: If $|V| \neq |W|$, then there are none. If $|V| = |W|$, then any bijection from V to W is an isomorphism. The same holds for empty graphs.

1.1.2. Path and cycle graphs

Next come two families of graphs with fairly simple shapes:

Definition 1.1.2. For each $n \in \mathbb{N}$, we define the ***n*-th path graph** P_n to be the simple graph

$$\begin{aligned}&(\{1,2,\ldots,n\}, \ \ \{\{i,i+1\} \ \mid \ 1 \leq i < n\})\\ &= (\{1,2,\ldots,n\}, \ \ \{12, \ 23, \ 34, \ \ldots, \ (n-1)\,n\}).\end{aligned}$$

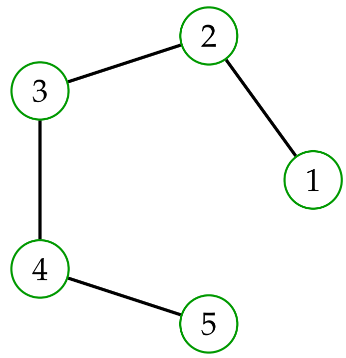
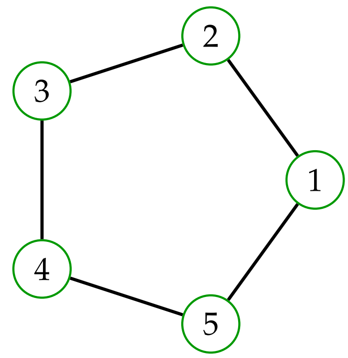
This graph has n vertices and $n-1$ edges (unless $n=0$, in which case it has 0 edges).

Definition 1.1.3. For each $n > 1$, we define the **n-th cycle graph** C_n to be the simple graph

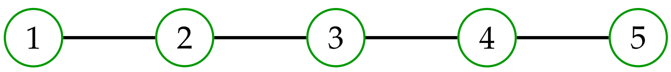
$$\begin{aligned}&(\{1,2,\ldots,n\}, \ \{\{i,i+1\} \mid 1 \leq i < n\} \cup \{\{n,1\}\}) \\ &= (\{1,2,\ldots,n\}, \ \{12, \ 23, \ 34, \ \ldots, \ (n-1)\,n, \ n1\}).\end{aligned}$$

This graph has n vertices and n edges (unless $n=2$, in which case it has 1 edge only). (We will later modify the definition of the 2-nd cycle graph C_2 somewhat, in order to force it to have 2 edges. But we cannot do this yet, since a simple graph with 2 vertices cannot have 2 edges.)

The following pictures show the path graph P_5 and the cycle graph C_5:

path graph	cycle graph
	

Of course, it is more common to draw the path graph stretched out horizontally:



Note that the cycle graph C_3 is identical with the complete graph K_3.

Question: What are the graph isomorphisms from P_n to itself?

Answer: One such isomorphism is the identity map $\mathrm{id} : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$. Another is the "reversal" map

$$\begin{aligned}\{1,2,\ldots,n\} &\to \{1,2,\ldots,n\}, \\ i &\mapsto n+1-i.\end{aligned}$$

There are no others.

Question: What are the graph isomorphisms from C_n to itself?

Answer: For any $k \in \mathbb{Z}$, we can define a "rotation by k vertices", which is the map

$$\begin{aligned}&\{1,2,\ldots,n\} \to \{1,2,\ldots,n\}, \\ &\qquad i \mapsto (i+k \text{ reduced modulo } n \text{ to an element of } \{1,2,\ldots,n\}).\end{aligned}$$

Thus we get n rotations (one for each $k \in \{1, 2, \ldots, n\}$); all of them are graph isomorphisms.

There are also the reflections, which are the maps

$$\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\},$$
$$i \mapsto (k - i \text{ reduced modulo } n \text{ to an element of } \{1, 2, \ldots, n\})$$

for $k \in \mathbb{Z}$. There are n of them, too, and they are isomorphisms as well.

Altogether we obtain $2n$ isomorphisms (for $n > 2$), and there are no others. (The group they form is the n-th dihedral group.)

1.1.3. Kneser graphs

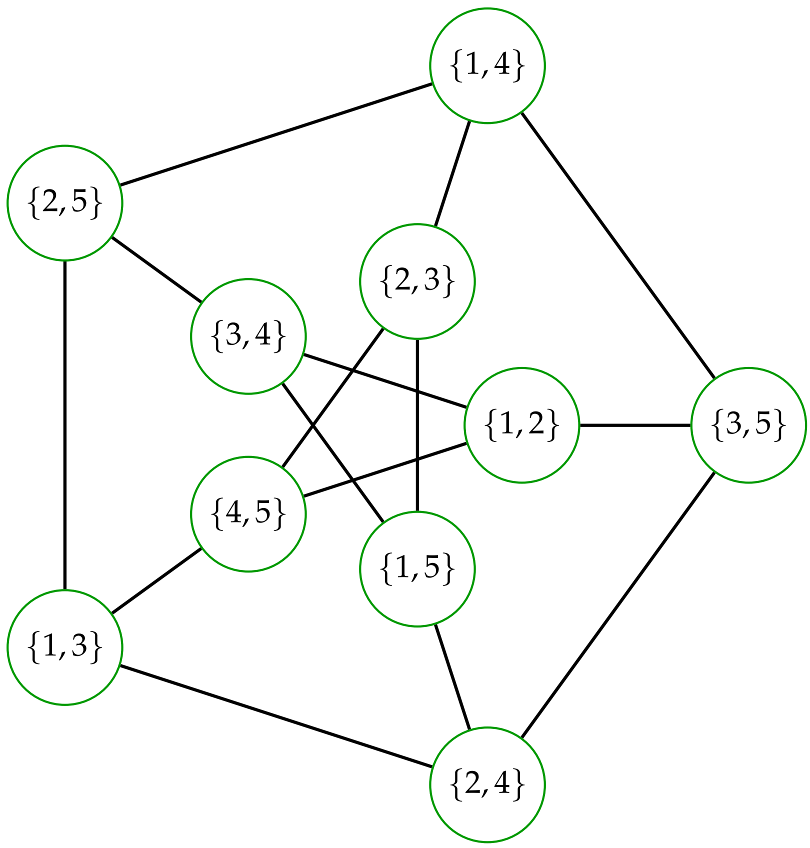
Here is a more obscure family of graphs:

Example 1.1.4. If S is a finite set, and if $k \in \mathbb{N}$, then we define the ***k*-th Kneser graph of** S to be the simple graph

$$K_{S,k} := (\mathcal{P}_k(S), \ \{IJ \mid I, J \in \mathcal{P}_k(S) \text{ and } I \cap J = \varnothing\}).$$

The vertices of $K_{S,k}$ are the k-element subsets of S, and two such subsets are adjacent if they are disjoint.

The graph $K_{\{1,2,\ldots,5\},2}$ is called the **Petersen graph**; here is how it looks like:



1.2. Subgraphs

Definition 1.2.1. Let $G = (V, E)$ be a simple graph.

(a) A **subgraph** of G means a simple graph of the form $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. In other words, a subgraph of G means a simple graph whose vertices are vertices of G and whose edges are edges of G.

(b) Let S be a subset of V. The **induced subgraph of** G **on the set** S denotes the subgraph

$$(S, \quad E \cap \mathcal{P}_2(S))$$

of G. In other words, it denotes the subgraph of G whose vertices are the elements of S, and whose edges are precisely those edges of G whose both endpoints belong to S.

(c) An **induced subgraph** of G means a subgraph of G that is the induced subgraph of G on S for some $S \subseteq V$.

Thus, a subgraph of a graph G is obtained by throwing away some vertices and some edges of G (in such a way, of course, that no edges remain "dangling" – i.e., if you throw away a vertex, then you must throw away all edges that contain this vertex). Such a subgraph is an induced subgraph if no edges are removed without need – i.e., if you removed only those edges that lost some of their endpoints. Thus, induced subgraphs can be characterized as follows:

Proposition 1.2.2. Let H be a subgraph of a simple graph G. Then, H is an induced subgraph of G if and only if each edge uv of G whose endpoints u and v belong to $\operatorname{V}(H)$ is an edge of H.

Proof. This is a matter of understanding the definition. $\square$

Example 1.2.3. Let $n > 1$ be an integer.

(a) The path graph P_n is a subgraph of the cycle graph C_n. It is not an induced subgraph (for $n > 2$), because it contains the two vertices n and 1 of C_n but does not contain the edge $n1$.

(b) The path graph P_{n-1} is an induced subgraph of P_n. (Namely, it is the induced subgraph of P_n on the set $\{1, 2, \ldots, n-1\}$.)

(c) Assume that $n > 3$. Is C_{n-1} a subgraph of C_n ? No, because the edge $(n-1)\,1$ belongs to C_{n-1} but not to C_n.

The following is easy:

Proposition 1.2.4. Let G be a simple graph, and let H be a subgraph of G. Assume that H is a complete graph. Then, H is automatically an induced subgraph of G.

Proof. This follows from the preceding proposition, since the completeness of H means that each 2-element subset $\{u, v\}$ of the vertex set of H is an edge of H. □

We note that triangles in a graph can be characterized in terms of complete subgraphs. Namely, a triangle "is" the same as a complete subgraph (or, equivalently, induced complete subgraph) with three vertices:

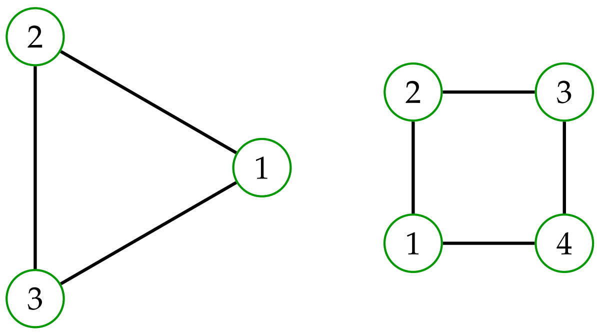
Remark 1.2.5. Let G be a simple graph. Let u, v, w be three distinct vertices of G. The following are equivalent:

1. The set $\{u, v, w\}$ is a triangle of G.
2. The induced subgraph of G on $\{u, v, w\}$ is isomorphic to K_3.
3. The induced subgraph of G on $\{u, v, w\}$ is isomorphic to C_3.

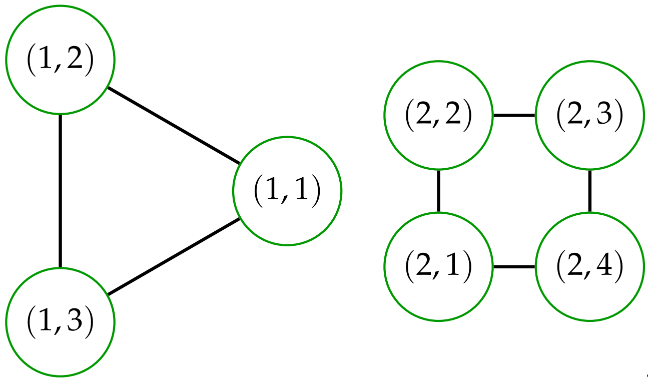
Thus, instead of saying "triangle of G", one often says "a K_3 in G" or "a C_3 in G". Generally, "an H in G" (where H and G are two graphs) means a subgraph of G that is isomorphic to H. (In the case when $H = K_3 = C_3$, it does not matter whether we require it to be a subgraph or an induced subgraph, since a complete subgraph has to be induced automatically.)

1.3. Disjoint unions

Another way of constructing new graphs from old is the disjoint union. The idea is simple: Taking the disjoint union $G_1 \sqcup G_2 \sqcup \cdots \sqcup G_k$ of several simple graphs $G_1, G_2, \ldots, G_k$ means putting the graphs alongside each other and treating the result as one big graph. To make this formally watertight, we have to relabel each vertex v of each graph G_i as the pair (i, v), so that vertices coming from different graphs appear as different even if they were equal. For example, the disjoint union $C_3 \sqcup C_4$ of the two cycle graphs C_3 and C_4 should not be



(which makes no sense, because there are two points labelled 1 in this picture, but a graph can have only one vertex 1), but rather should be



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So here is the formal definition:

Definition 1.3.1. Let $G_1, G_2, \ldots, G_k$ be simple graphs, where $G_i = (V_i, E_i)$ for each $i \in \{1, 2, \ldots, k\}$. The **disjoint union** of these k graphs $G_1, G_2, \ldots, G_k$ is defined to be the simple graph (V, E), where

$$V = \{(i, v) \mid i \in \{1, 2, \ldots, k\} \text{ and } v \in V_i\} \qquad \text{and}$$
$$E = \{\{(i, v_1), (i, v_2)\} \mid i \in \{1, 2, \ldots, k\} \text{ and } \{v_1, v_2\} \in E_i\}.$$

This disjoint union is denoted by $G_1 \sqcup G_2 \sqcup \cdots \sqcup G_k$.

Note: If G and H are two graphs, then the two graphs $G \sqcup H$ and $H \sqcup G$ are isomorphic, but not the same graph (unless $G = H$). For example, $C_3 \sqcup C_4$ has a vertex $(2, 4)$, but $C_4 \sqcup C_3$ does not.

1.4. Walks and paths

We now come to the definitions of walks and paths – two of the most fundamental features that graphs can have. In particular, Euler's 1736 paper, where graphs were first studied, is about certain kinds of walks.

1.4.1. Definitions

Imagine a graph as a road network, where each vertex is a town and each edge is a (bidirectional) road. By successively walking along several edges, you can often get from a town to another even if they are not adjacent. This is made formal in the concept of a "walk":

Definition 1.4.1. Let G be a simple graph. Then:

(a) A **walk** (in G) means a finite sequence $(v_0, v_1, \ldots, v_k)$ of vertices of G (with $k \geq 0$) such that all of v_0v_1, v_1v_2, v_2v_3, $\ldots$, $v_{k-1}v_k$ are edges of G. (The latter condition is vacuously true if $k = 0$.)

(b) If $\mathbf{w} = (v_0, v_1, \ldots, v_k)$ is a walk in G, then:

- The **vertices** of $\mathbf{w}$ are defined to be $v_0, v_1, \ldots, v_k$.
- The **edges** of $\mathbf{w}$ are defined to be $v_0v_1,\ v_1v_2,\ v_2v_3,\ \ldots,\ v_{k-1}v_k$.
- The nonnegative integer k is called the **length** of $\mathbf{w}$. (This is the number of all edges of $\mathbf{w}$, counted with multiplicity. It is 1 smaller than the number of all vertices of $\mathbf{w}$, counted with multiplicity.)
- The vertex v_0 is called the **starting point** of $\mathbf{w}$. We say that $\mathbf{w}$ **starts** (or **begins**) at v_0.
- The vertex v_k is called the **ending point** of $\mathbf{w}$. We say that $\mathbf{w}$ **ends** at v_k.

(c) A **path** (in G) means a walk (in G) whose vertices are distinct. In other words, a path means a walk $(v_0, v_1, \ldots, v_k)$ such that $v_0, v_1, \ldots, v_k$ are distinct.

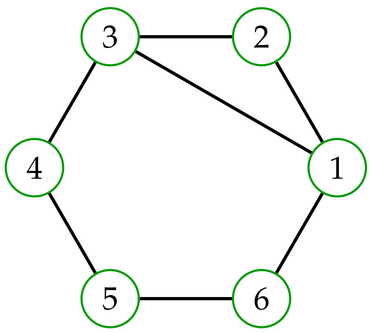
(d) Let p and q be two vertices of G. A **walk from** p **to** q means a walk that starts at p and ends at q. A **path from** p **to** q means a path that starts at p and ends at q.

(e) We often say "walk of G" and "path of G" instead of "walk in G" and "path in G", respectively.

Example 1.4.2. Let G be the graph

$$(\{1,2,3,4,5,6\},\ \ \{12,\ 23,\ 34,\ 45,\ 56,\ 61,\ 13\}).$$

This graph looks as follows:



Then:

- The sequence $(1,3,4,5,6,1,3,2)$ of vertices of G is a walk in G. This walk is a walk from 1 to 2. It is not a path. The length of this walk is 7.

- The sequence $(1,2,4,3)$ of vertices of G is not a walk, since 24 is not an edge of G. Hence, it is not a path either.

- The sequence $(1,3,2,1)$ is a walk from 1 to 1. It has length 3. It is not a path.
- The sequence $(1,2,1)$ is a walk from 1 to 1. It has length 2. It is not a path.
- The sequence (5) is a walk from 5 to 5. It has length 0. It is a path. More generally, each vertex v of G produces a length-0 path (v).
- The sequence $(5,4)$ is a walk from 5 to 4. It has length 1. It is a path. More generally, each edge uv of G produces a length-1 path (u,v).

Exercise 1. Prove that the edges of a path are always distinct. (See HW1 in Spring 2017 for a rigorous proof.)

1.4.2. Composing/concatenating and reversing walks

Here are some simple things we can do with walks and paths.

First, we can "splice" two walks together if the ending point of the first is the starting point of the second:

Proposition 1.4.3. Let G be a simple graph. Let u, v and w be three vertices of G. Let $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ be a walk from u to v. Let $\mathbf{b} = (b_0, b_1, \ldots, b_\ell)$ be a walk from v to w. Then,

$$\begin{aligned}(a_0, a_1, \ldots, a_k, b_1, b_2, \ldots, b_\ell) &= (a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_\ell) \\ &= (a_0, a_1, \ldots, a_{k-1}, v, b_1, b_2, \ldots, b_\ell)\end{aligned}$$

is a walk from u to w. This walk shall be denoted $\mathbf{a} * \mathbf{b}$.

Proof. Intuitively clear and straightforward to verify. □

Proposition 1.4.4. Let G be a simple graph. Let u and v be two vertices of G. Let $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ be a walk from u to v. Then:

(a) The list $(a_k, a_{k-1}, \ldots, a_0)$ is a walk from v to u. We denote this walk by $\operatorname{rev} \mathbf{a}$ and call it the **reversal** of $\mathbf{a}$.

(b) If $\mathbf{a}$ is a path, then $\operatorname{rev} \mathbf{a}$ is a path again.

Proof. Intuitively clear and straightforward to verify. □

1.4.3. Reducing walks to paths

A path is just a walk without repeated vertices. If you have a walk, you can turn it into a path by removing "loops" (or "digressions"):

Proposition 1.4.5. Let G be a simple graph. Let u and v be two vertices of G. Let $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ be a walk from u to v. Assume that $\mathbf{a}$ is not a path. Then, there exists a walk from u to v whose length is smaller than k.

Proof. Since $\mathbf{a}$ is not a path, two of its vertices are equal. In other words, there exist $i < j$ such that $a_i = a_j$. Consider these i and j. Now, consider the tuple

$$\left(\underbrace{a_0, a_1, \ldots, a_i}_{\text{the first } i+1 \text{ vertices of } \mathbf{a}}, \underbrace{a_{j+1}, a_{j+2}, \ldots, a_k}_{\text{the last } k-j \text{ vertices of } \mathbf{a}} \right)$$

(this is just $\mathbf{a}$ with the part between a_i and a_j cut out). This tuple is a walk from u to v, and its length is $\underbrace{i}_{<j} + (k-j) < j + (k-j) = k$. So we have found a walk from u to v whose length is smaller than k. This proves the proposition. □

Example 1.4.6. Consider the walk $(1, 3, 4, 5, 6, 1, 3, 2)$ from Example 1.4.2. Then, Proposition 1.4.5 tells us that there is a walk from 1 to 2 that has smaller length. You can find this walk by removing the part between the two 3's. You get the walk $(1, 3, 2)$. This is actually a path.

Corollary 1.4.7 (When there is a walk, there is a path)**.** Let G be a simple graph. Let u and v be two vertices of G. Assume that there is a walk from u to v of length k for some $k \in \mathbb{N}$. Then, there is a path from u to v of length $\leq k$.

Proof. Apply Proposition 1.4.5 several times, until you get a path. (You will eventually get a path, because the length cannot keep decreasing forever.) □