Math 530 Spring 2022, Lecture 25: Applications of Hall's marriage theorem

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Matchings (cont'd)

1.1. Hall's marriage theorem (repeated)

We stated Hall's marriage theorem last time:

Theorem 1.1.1 (Hall's marriage theorem, short: HMT)**.** Let (G, X, Y) be a bipartite graph. Assume that each subset A of X satisfies $|N(A)| \geq |A|$. (This assumption is called the "**Hall condition**".)

Then, G has an X-complete matching.

We won't prove this now (this is what Lecture 26 will be for), but we will give several more applications.

1.2. Systems of representatives

There are two more equivalent form of the HMT that have the "advantage" that they do not rely on the notion of a graph. When non-combinatorialists use the HMT, they often use it in one of these forms. Here is the first form:

Theorem 1.2.1 (existence of SDR)**.** Let $A_1, A_2, \ldots, A_n$ be any n sets. Assume that the union of any p of these sets has size $\geq p$, for all $p \in \{0, 1, \ldots, n\}$. (In other words, assume that

$$\left|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_p}\right| \geq p \qquad \text{for any } 1 \leq i_1 < i_2 < \cdots < i_p \leq n.$$

)

Then, we can find n **distinct** elements

$$a_1 \in A_1, \qquad a_2 \in A_2, \qquad \ldots, \qquad a_n \in A_n.$$

Remark 1.2.2. An n-tuple $(a_1, a_2, \ldots, a_n)$ of n distinct elements like this is called a **system of distinct representatives** for our n sets $A_1, A_2, \ldots, A_n$. (This is often abbreviated "SDR".)

Example 1.2.3. Take a standard deck of cards, and deal them out into 13 piles of 4 cards each – e.g., as follows:

$$\{2\spadesuit,\ 2\heartsuit,\ 9\diamondsuit,\ K\diamondsuit\}, \quad \{A\spadesuit,\ A\heartsuit,\ 3\spadesuit,\ 3\diamondsuit\}, \quad \{A\diamondsuit,\ 4\clubsuit,\ 5\clubsuit,\ Q\clubsuit\},$$
$$\{2\diamondsuit,\ 4\heartsuit,\ 5\heartsuit,\ 5\spadesuit\}, \quad \{A\clubsuit,\ 7\clubsuit,\ 7\spadesuit,\ 7\heartsuit\}, \quad \{4\spadesuit,\ 6\spadesuit,\ 6\diamondsuit,\ 6\clubsuit\},$$
$$\{3\heartsuit,\ 3\clubsuit,\ 8\spadesuit,\ 8\heartsuit\}, \quad \{2\clubsuit,\ K\clubsuit,\ K\heartsuit,\ 10\heartsuit\}, \quad \{4\diamondsuit,\ 5\diamondsuit,\ 9\spadesuit,\ 9\heartsuit\},$$
$$\{Q\spadesuit,\ Q\heartsuit,\ Q\diamondsuit,\ Q\clubsuit\}, \quad \{6\heartsuit,\ J\spadesuit,\ J\diamondsuit,\ J\clubsuit\}, \quad \{7\diamondsuit,\ 8\diamondsuit,\ 8\clubsuit,\ 9\clubsuit\},$$
$$\{10\spadesuit,\ J\heartsuit,\ 10\diamondsuit,\ 10\clubsuit\}$$

(you can distribute the cards among the piles randomly; this is just one example). Then, I claim that it is possible to select exactly 1 card from each pile so that the 13 selected cards contain exactly 1 card of each rank (i.e., exactly one ace, exactly one 2, exactly one 3, and so on).

Indeed, this follows from Theorem 1.2.1 (applied to $A_i = \{\text{ranks that occur in the } i\text{-th pile}\}$) because any p piles contain cards of at least p different ranks.

Proof of Theorem 1.2.1. First, we WLOG assume that all n sets $A_1, A_2, \ldots, A_n$ are finite. (If not, then we can just replace each infinite one by an n-element subset thereof. The assumption $\left|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_p}\right| \geq p$ will not be disturbed by this change – make sure you understand why!)

Furthermore, we WLOG assume that no integer belongs to any of the n sets $A_1, A_2, \ldots, A_n$ (otherwise, we just rename the elements of these sets so that they aren't integers any more).

Now, let $X = \{1, 2, \ldots, n\}$ and $Y = A_1 \cup A_2 \cup \cdots \cup A_n$. Both sets X and Y are finite, and are disjoint.

We define a simple graph G as follows:

- The vertices of G are the elements of $X \cup Y$.
- A vertex $x \in X$ is adjacent to a vertex $y \in Y$ if and only if $y \in A_x$. There are no further adjacencies.

Thus, (G, X, Y) is a bipartite graph. The assumption $\left|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_p}\right| \geq p$ ensures that it satisfies the Hall condition. Hence, by the HMT (Theorem 1.1.1), we conclude that this graph G has an X-complete matching. This matching must have the form

$$\{\{1, a_1\},\ \{2, a_2\},\ \ldots,\ \{n, a_n\}\}$$

for some $a_1, a_2, \ldots, a_n \in Y$ (since (G, X, Y) is bipartite, so that the partners of the vertices $1, 2, \ldots, n \in X$ must belong to Y). These elements $a_1, a_2, \ldots, a_n \in Y$ are distinct (since two edges in a matching cannot have a common endpoint), and each $i \in \{1, 2, \ldots, n\}$ satisfies $a_i \in A_i$ (since the vertex a_i is adjacent to i in

G). Thus, these $a_1, a_2, \ldots, a_n$ are precisely the n distinct elements we are looking for. This proves Theorem 1.2.1. □

Conversely, it is not hard to derive the HMT from Theorem 1.2.1.

Here is the second set-theoretical restatement of the HMT:

Theorem 1.2.4 (existence of SCR)**.** Let $A_1, A_2, \ldots, A_n$ be n sets. Let $B_1, B_2, \ldots, B_m$ be m sets. Assume that for any numbers $1 \leq i_1 < i_2 < \cdots < i_p \leq n$, there exist at least p elements $j \in \{1, 2, \ldots, m\}$ such that the union $A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_p}$ has nonempty intersection with B_j. Then, there exists an injective map $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ such that all $i \in \{1, 2, \ldots, n\}$ satisfy $A_i \cap B_{\sigma(i)} \neq \varnothing$.

Proof. We leave this to the reader. Again, construct an appropriate bipartite graph and apply the HMT. □

(The "SCR" in the name of the theorem is short for "system of common representatives".)

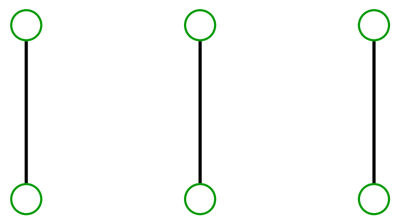
See [MirPer66] for much more about systems of representatives.

1.3. Regular bipartite graphs

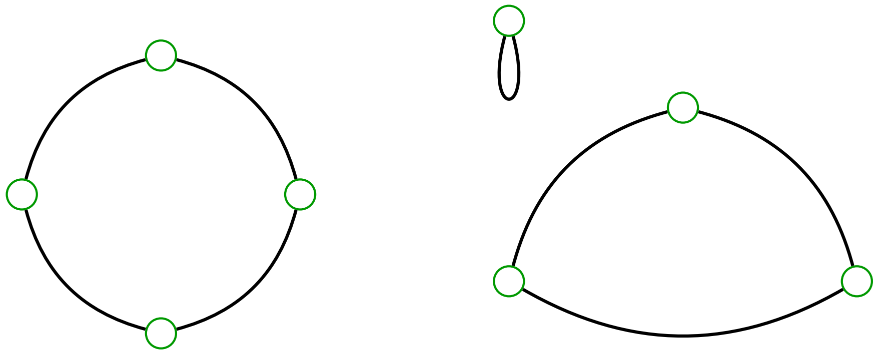
The HMT gives a necessary and sufficient criterion for the existence of an X-complete matching in an arbitrary bipartite graph. In the more restrictive setting of **regular** bipartite graphs – i.e., bipartite graphs where each vertex has the same degree –, there is a simpler sufficient condition: such a matching always exists! We shall soon prove this surprising fact (which is not hard using the HMT), but first let us get the definition in order:

Definition 1.3.1. Let $k \in \mathbb{N}$. A multigraph G is said to be k**-regular** if all its vertices have degree k.

Example 1.3.2. A 1-regular graph is a graph whose entire edge set is a perfect matching. In other words, a 1-regular graph is a graph that is a disjoint union of copies of the 2-nd path graph P_2. Here is an example of such a graph:

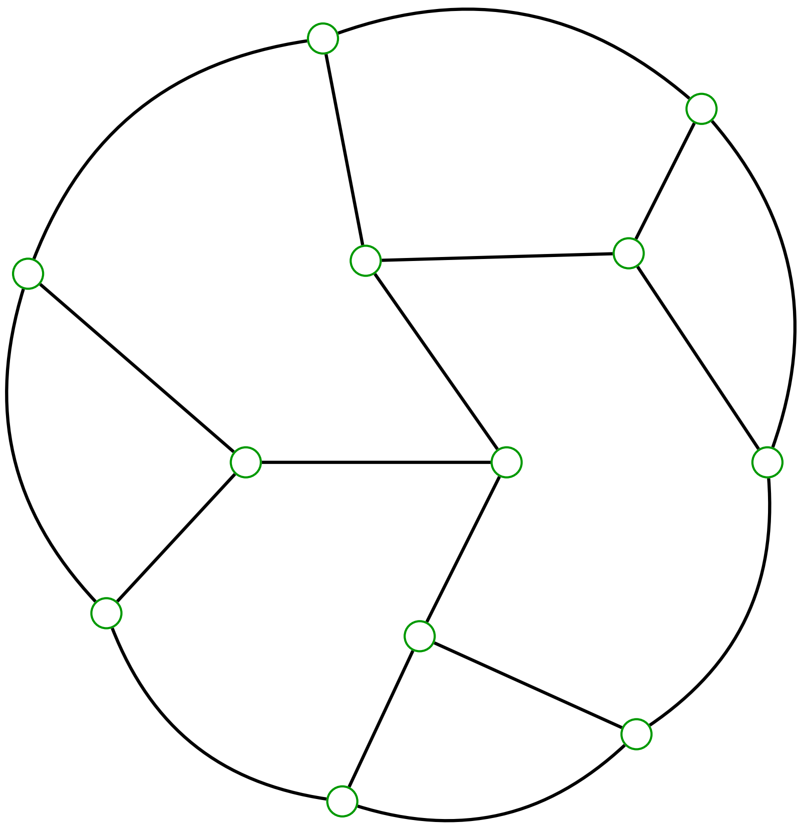


Example 1.3.3. A 2-regular graph is a graph that is a disjoint union of cycle graphs. Here is an example of such a graph:



(yes, a C_1 is fine, and so would be a C_2).

Example 1.3.4. The 3-regular graphs are known as **cubic graphs** or **trivalent graphs**. An example is the Petersen graph. Here is another example (known as the **Frucht graph**):



More examples of cubic graphs can be found on the Wikipedia page. There is no hope of describing them all.

Example 1.3.5. Any Kneser graph $K_{S,k}$ is $\dbinom{|S|-k}{k}$-regular.

Proof. This is saying that if A is a k-element subset of a finite set S, then there are precisely $\dbinom{|S|-k}{k}$ many k-element subsets of S that are disjoint from A. But this is clear, since the latter subsets are just the k-element subsets of the $(|S|-k)$-element set $S \setminus A$. $\square$

Proposition 1.3.6. Let $k > 0$. Let (G, X, Y) be a k-regular bipartite graph (i.e., a bipartite graph such that G is k-regular). Then, $|X| = |Y|$.

Proof. Write the multigraph G as $G = (V, E, \varphi)$. Each edge $e \in E$ contains exactly one vertex $x \in X$ (since (G, X, Y) is a bipartite graph). Hence,

$$|E| = \sum_{x\in X} \underbrace{(\text{\# of edges that contain the vertex } x)}_{=\deg x} = \sum_{x\in X} \underbrace{\deg x}_{\substack{=k \\ (\text{since } G \text{ is } k\text{-regular})}}$$
$$= \sum_{x\in X} k = k \cdot |X| .$$

Similarly, $|E| = k \cdot |Y|$. Comparing these two equalities, we obtain $k \cdot |X| = k \cdot |Y|$. Since $k > 0$, we can divide this by k, and conclude $|X| = |Y|$. $\square$

Theorem 1.3.7 (Frobenius matching theorem)**.** Let $k > 0$. Let (G, X, Y) be a k-regular bipartite graph (i.e., a bipartite graph such that G is k-regular). Then, G has a perfect matching.

Proof. First, we claim that each subset A of X satisfies $|N(A)| \geq |A|$.

Indeed, let A be a subset of X. Consider the edges of G that have at least one endpoint in A. We shall call such edges "A-edges". How many A-edges are there?

On the one hand, each A-edge contains exactly one vertex in A (why?[1]). Thus,

$$(\text{\# of } A\text{-edges}) = \sum_{x\in A} \underbrace{(\text{\# of } A\text{-edges containing the vertex } x)}_{\substack{=\deg x \\ (\text{since each edge that contains the vertex } x \\ \text{is an } A\text{-edge})}}$$
$$= \sum_{x\in A} \underbrace{\deg x}_{\substack{=k \\ (\text{since } G \text{ is } k\text{-regular})}} = \sum_{x\in A} k = k \cdot |A| .$$

[1]Here we are using the fact that $A \subseteq X$, so that no two vertices in A can be adjacent.

On the other hand, each A-edge contains exactly one vertex in $N(A)$ (why?[2]). Thus,

$$\begin{aligned}(\#\text{ of }A\text{-edges}) &= \sum_{y\in N(A)} \underbrace{(\#\text{ of }A\text{-edges containing the vertex }y)}_{\leq \deg y} \\ &\leq \sum_{y\in N(A)} \underbrace{\deg y}_{\substack{=k \\ (\text{since }G\text{ is }k\text{-regular})}} = \sum_{y\in N(A)} k = k\cdot |N(A)| .\end{aligned}$$

Hence,

$$k\cdot |N(A)| \geq (\#\text{ of }A\text{-edges}) = k\cdot |A| .$$

Since $k > 0$, we can divide this inequality by k, and thus find $|N(A)| \geq |A|$.

Forget that we fixed A. We thus have proved $|N(A)| \geq |A|$ for each subset A of X. Hence, the HMT (Theorem 1.1.1) yields that the graph G has an X-complete matching M. Consider this M.

However, Proposition 1.3.6 yields $|X| = |Y|$. Hence, Proposition 1.3.1 **(f)** in Lecture 24 shows that the matching M is perfect (since M is X-complete). Therefore, G has a perfect matching. This proves Theorem 1.3.7. □

1.4. Latin squares

One of many applications of Theorem 1.3.7 is to the study of Latin squares. Here is the definition of this concept:

Definition 1.4.1. Let $n \in \mathbb{N}$. A **Latin square** of order n is an $n \times n$-matrix M that satisfies the following conditions:

1. The entries of M are the numbers $1, 2, \ldots, n$, each appearing exactly n times.

2. In each row of M, the entries are distinct.

3. In each column of M, the entries are distinct.

Example 1.4.2. Here is a Latin square of order 5:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

[2] Here we are using the fact that $N(A) \subseteq Y$ (which follows from $A \subseteq X$ using Proposition 1.2.8 in Lecture 24), so that no two vertices in $N(A)$ can be adjacent.

Similarly, for each $n \in \mathbb{N}$, the matrix $\left(c_{i+j-1}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n}$, where

$$c_k = \begin{cases} k, & \text{if } k \leq n; \\ k-n, & \text{else,} \end{cases}$$

is a Latin square of order n.

A popular example of Latin squares of order 9 are Sudokus (but they have to satisfy an additional requirement, concerning certain 3×3 subsquares).

The Latin squares in Example 1.4.2 are rather boring. What would be a good algorithm to construct general Latin squares?

Here is an attempt at a recursive algorithm: We just start by filling in the first row, then the second row, then the third row, and so on, making sure at each step that the distinctness conditions (Conditions 2 and 3 in Definition 1.4.1) are satisfied.

Example 1.4.3. Let us construct a Latin square of order 5 by this algorithm. We begin (e.g.) with the first row

$$\left(\begin{array}{ccccc} 3 & 1 & 4 & 2 & 5 \end{array}\right).$$

Then, we append a second row $\left(\begin{array}{ccccc} 2 & 4 & 1 & 5 & 3 \end{array}\right)$ to it, chosen in such a way that its five entries are distinct and also each entry is distinct from the entry above (again, there are many possibilities; we have just picked one). Thus, we have our first two rows:

$$\left(\begin{array}{ccccc} 3 & 1 & 4 & 2 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{array}\right).$$

We continue along the same lines, ending up with the Latin square

$$\left(\begin{array}{ccccc} 3 & 1 & 4 & 2 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 1 & 5 & 2 & 3 & 4 \\ 5 & 2 & 3 & 4 & 1 \\ 4 & 3 & 5 & 1 & 2 \end{array}\right)$$

(or another, depending on the choices we have made).

Does this algorithm always work?

To be fully honest, it's not a fully specified algorithm, since I haven't explained how to fill a row (it's not straightforward). But let's assume that we know how to do this, if it is at all possible. The natural question is: Will we always be able to produce a complete Latin square using this algorithm, or will we get stuck somewhere (having constructed k rows for some $k < n$, but being unable to produce a $(k+1)$-st row)?

It turns out that we won't get stuck this way. In other words, the following holds:

Proposition 1.4.4. Let $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n-1\}$. Then, any $k \times n$ **Latin rectangle** (i.e., any $k \times n$-matrix that contains the entries $1, 2, \ldots, n$, each appearing exactly k times, and satisfies the Conditions 2 and 3 from Definition 1.4.1) can be extended to a $(k+1) \times n$ Latin rectangle by adding an appropriately chosen extra row at the bottom.

Proof. Let M be a $k \times n$ Latin rectangle[3]. We want to find a new row that we can append to M at the bottom, such that the result will be a $(k+1) \times n$ Latin rectangle.

This new row should contain the numbers $1, 2, \ldots, n$ in some order. Moreover, for each $i \in \{1, 2, \ldots, n\}$, its i-th entry should be distinct from all entries of the i-th column of M. How do we find such a new row?

Let $X = \{1, 2, \ldots, n\}$ and $Y = \{-1, -2, \ldots, -n\}$.

Let G be the simple graph with vertex set $X \cup Y$, where we let a vertex $i \in X$ be adjacent to a vertex $-j \in Y$ if and only if the number j does not appear in the i-th column of M. There should be no further adjacencies.

Thus, (G, X, Y) is a bipartite graph. Moreover, the graph G is $(n-k)$-regular (this is not hard to see[4]). Thus, by the Frobenius matching theorem (Theorem 1.3.7), the graph G has a perfect matching. Let

$$\{\{1, \ -a_1\}, \{2, \ -a_2\}, \ \ldots, \ \{n, \ -a_n\}\}$$

be this perfect matching. Then, the numbers $a_1, a_2, \ldots, a_n$ are distinct (since two edges in a matching cannot have a common endpoint), and the number a_i does not appear in the i-th column of M (since $\{i, \ -a_i\}$ is an edge of G). Thus, we can append the row

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

to M at the bottom and obtain a $(k+1) \times n$ Latin rectangle. This proves Proposition 1.4.4. $\square$

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[3]For example, if $n = 5$ and $k = 3$, then M can be $\begin{pmatrix} 3 & 1 & 4 & 2 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 1 & 5 & 2 & 3 & 4 \end{pmatrix}$.

[4]*Proof.* Each vertex $i \in X$ has degree $n-k$ (after all, there are k numbers in $\{1, 2, \ldots, n\}$ that appear in the i-th column of M, thus $n-k$ numbers in $\{1, 2, \ldots, n\}$ that **do not** appear in this column). It remains to show that each vertex $-j \in Y$ has degree $n-k$ as well. To see this, consider some vertex $-j \in Y$. Then, the number j appears exactly once in each row of M (since Condition 2 forces each row to contain the numbers $1, 2, \ldots, n$ in some order). Hence, the number j appears a total of k times in M. These k appearances of j must be in k distinct columns (since having two of them in the same column would conflict with Condition 3). Thus, there are k columns of M that contain j, and therefore $n-k$ columns that don't. In other words, the vertex $-j \in Y$ has degree $n-k$.

1.5. Magic matrices and the Birkhoff–von Neumann theorem

Let us now apply the HMT to linear algebra.

Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$. We also set $\mathbb{R}_+ := \{\text{all nonnegative reals}\}$.

Here are three very similar definitions:

Definition 1.5.1. An **$\mathbb{N}$-magic matrix** means an $n \times n$-matrix M that satisfies the following three conditions:

1. All entries of M are nonnegative integers.
2. The sum of the entries in each row of M is equal.
3. The sum of the entries in each column of M is equal.

Definition 1.5.2. An **$\mathbb{R}_+$-magic matrix** means an $n \times n$-matrix M that satisfies the following three conditions:

1. All entries of M are nonnegative reals.
2. The sum of the entries in each row of M is equal.
3. The sum of the entries in each column of M is equal.

Definition 1.5.3. A **doubly stochastic matrix** means an $n \times n$-matrix M that satisfies the following three conditions:

1. All entries of M are nonnegative reals.
2. The sum of the entries in each row of M is 1.
3. The sum of the entries in each column of M is 1.

Clearly, these three concepts are closely related (in particular, all $\mathbb{N}$-magic matrices and all doubly stochastic matrices are $\mathbb{R}_+$-magic). The most important of them is the last; in particular, majorization theory (one of the main methods for proving inequalities) is deeply connected to the properties of doubly stochastic matrices (see [MaOlAr11, Chapter 2]). See [BapRag97, Chapter 2] for a chapter-length treatment of doubly stochastic matrices. We shall only prove some of their most basic properties. First, some examples:

Example 1.5.4. For any $n > 0$, the $n \times n$-matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

is $\mathbb{N}$-magic and also $\mathbb{R}_+$-magic. This matrix is not doubly stochastic (unless $n = 1$), since the sum of the entries in a row or column is n, not 1. However, if we divide this matrix by n, it becomes doubly stochastic.

Example 1.5.5. Here is an $\mathbb{N}$-magic 3×3-matrix:

$$\begin{pmatrix} 7 & 0 & 5 \\ 2 & 6 & 4 \\ 3 & 6 & 3 \end{pmatrix}.$$

Dividing this matrix by 12 gives a doubly stochastic matrix.

Example 1.5.6. A **permutation matrix** is an $n \times n$-matrix whose entries are 0's and 1's, and which has exactly one 1 in each row and exactly one 1 in each column. For example, $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is a permutation matrix of size 4.

For any $n \in \mathbb{N}$, there are $n!$ many permutation matrices (of size n), since they are in bijection with the permutations of $\{1, 2, \ldots, n\}$. Namely, if σ is a permutation of $\{1, 2, \ldots, n\}$, then the corresponding permutation matrix $P(\sigma)$ has its $(i, \sigma(i))$-th entries equal to 1 for all $i \in \{1, 2, \ldots, n\}$, while its remaining $n^2 - n$ entries are 0. For example, if σ is the permutation of $\{1, 2, 3\}$ sending $1, 2, 3$ to $2, 3, 1$, then the corresponding permutation matrix $P(\sigma)$ is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Any permutation matrix is $\mathbb{N}$-magic, $\mathbb{R}_+$-magic and doubly stochastic.

It turns out that these permutation matrices are (in a sense) the "building blocks" of all magic (and doubly stochastic) matrices! Namely, the following holds:

Theorem 1.5.7 (Birkhoff–von Neumann theorem)**.** Let $n \in \mathbb{N}$. Then:

- **(a)** Any $\mathbb{N}$-magic $n \times n$-matrix can be expressed as a finite sum of permutation matrices.
- **(b)** Any $\mathbb{R}_+$-magic $n \times n$-matrix can be expressed as an $\mathbb{R}_+$-linear combination of permutation matrices (i.e., it can be expressed in the form $\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k$, where $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}_+$ are numbers and where $P_1, P_2, \ldots, P_k$ are permutation matrices).
- **(c)** Let $n > 0$. Any doubly stochastic $n \times n$-matrix can be expressed as a convex combination of permutation matrices (i.e., it can be expressed

in the form $\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k$, where $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}_+$ are numbers satisfying $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$ and where $P_1, P_2, \ldots, P_k$ are permutation matrices).

Soon we will sketch a proof of this theorem using the HMT. First, two simple results that will be used in the proof.

Proposition 1.5.8. Let A be an $\mathbb{N}$-magic or $\mathbb{R}_+$-magic $n \times n$-matrix. Then, the sum of all entries in a row of A equals the sum of all entries in a column of A.

Proof. Both sums equal $\dfrac{1}{n}$ times the sum of all entries of A (since A has n rows and n columns). □

Lemma 1.5.9. Let M be an $\mathbb{N}$-magic or $\mathbb{R}_+$-magic matrix that is not the zero matrix. Then, there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that all entries $M_{1,\sigma(1)}$, $M_{2,\sigma(2)}$, $\ldots$, $M_{n,\sigma(n)}$ are nonzero.

Example 1.5.10. If $n = 3$ and $M = \begin{pmatrix} 2 & 7 & 1 \\ 0 & 1 & 9 \\ 8 & 2 & 0 \end{pmatrix}$, then the permutation σ that sends $1, 2, 3$ to $3, 2, 1$ has this property.

Proof of Lemma 1.5.9. Let s denote the sum of the entries in any given row of M (it doesn't matter which row we take, since M is magic). Then, s is also the sum of the entries in any given column of M (by Proposition 1.5.8). Also, the sum of all entries of M is ns. Hence, $ns > 0$ (since M has nonnegative entries and is not the zero matrix). Thus, $s > 0$.

Let $X = \{1, 2, \ldots, n\}$ and $Y = \{-1, -2, \ldots, -n\}$.

Let G be the simple graph with vertex set $X \cup Y$ and with edges defined as follows: A vertex $i \in X$ shall be adjacent to a vertex $-j \in Y$ if and only if $M_{i,j} > 0$ (here, $M_{i,j}$ denotes the (i, j)-th entry of M). There shall be no further adjacencies.

Thus, (G, X, Y) is a bipartite graph.

We shall now prove that it satisfies the Hall condition. That is, we shall prove that every subset A of $\{1, 2, \ldots, n\}$ satisfies $|N(A)| \geq |A|$.

Assume the contrary. Thus, there exists a subset A of $\{1, 2, \ldots, n\}$ that satisfies $|N(A)| < |A|$. Consider this A. WLOG assume that $A = \{1, 2, \ldots, k\}$ for some $k \in \{0, 1, \ldots, n\}$ (otherwise, we permute the rows of M). Thus, all positive entries in the first k rows of A are concentrated in fewer than k columns (since the columns in which they lie are the j-th columns for $j \in N(A)$, but we have $|N(A)| < |A| = k$). Therefore, the sum of these entries is smaller than ks (since the sum of all entries in any given column is s). On the other hand, however, the sum of these entries equals ks, because they are all the positive entries in the first k rows of A (and the sum of all positive entries in a given row equals the sum of **all** entries in this row, which is s). The two preceding sentences clearly contradict each other. This contradiction shows that our assumption was false.

Hence, the Hall condition is satisfied. Thus, the HMT yields that G has a perfect matching. Let

$$\{\{1,-a_1\}, \{2,-a_2\}, \ldots, \{n,-a_n\}\}$$

be this perfect matching. Then, $a_1, a_2, \ldots, a_n$ are distinct, so we can find a permutation σ of $\{1,2,\ldots,n\}$ such that $a_i = \sigma(i)$ for all $i \in \{1,2,\ldots,n\}$. This permutation σ then satisfies $M_{i,\sigma(i)} > 0$ for all $i \in \{1,2,\ldots,n\}$, which is what we wanted. Thus, Lemma 1.5.9 is proved. □

Proof of Theorem 1.5.7 (sketched). **(a)** Let M be an $\mathbb{N}$-magic $n \times n$-matrix. How can we express M as a sum of permutation matrices?

We can try the following method: Try to subtract a permutation matrix from M in such a way that the result will still be an $\mathbb{N}$-magic matrix. Then do this again, and again and again... until we reach the zero matrix. Once we have arrived at the zero matrix, the sum of all the permutation matrices that we have subtracted along the way must be M.

Let us experience this method on an example: Let $n = 3$ and[5] $M = \begin{pmatrix} 2 & 7 & 1 \\ & 1 & 9 \\ 8 & 2 & \end{pmatrix}$. If we subtract a permutation matrix from M, then the resulting matrix will still satisfy Conditions 2 and 3 of Definition 1.5.1 (since the sum of the entries in any row has been decreased by 1, and the sum of the entries in any column has also been decreased by 1); however, Condition 1 is not guaranteed, since the subtraction may turn an entry of M negative (which is not allowed). For example, this would happen if we tried to subtract the permutation matrix $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ from M. Fortunately, Lemma 1.5.9 tells us that there is a permutation σ of $\{1,2,\ldots,n\}$ such that all entries $M_{1,\sigma(1)}, M_{2,\sigma(2)}, \ldots, M_{n,\sigma(n)}$ are nonzero. If we choose such a σ, and subtract the corresponding permutation matrix $P(\sigma)$ from M, then we obtain an $\mathbb{N}$-magic matrix, because subtracting 1 from the nonzero entries $M_{1,\sigma(1)}, M_{2,\sigma(2)}, \ldots, M_{n,\sigma(n)}$ cannot render any of these entries negative. In our example, we can pick σ to be the permutation that sends

[5]We are here omitting zero entries from matrices. Thus, $\begin{pmatrix} 2 & 7 & 1 \\ & 1 & 9 \\ 8 & 2 & \end{pmatrix}$ means the matrix $\begin{pmatrix} 2 & 7 & 1 \\ 0 & 1 & 9 \\ 8 & 2 & 0 \end{pmatrix}$.

1, 2, 3 to 3, 2, 1. The corresponding permutation matrix $P(\sigma)$ is $\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$.
Subtracting this matrix from M, we find

$$\begin{pmatrix} 2 & 7 & 1 \\ & 1 & 9 \\ 8 & 2 & \end{pmatrix} - \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} 2 & 7 & \\ & & 9 \\ 7 & 2 & \end{pmatrix}.$$

This is again an $\mathbb{N}$-magic matrix. Thus, let us do the same to it that we did to M: We again subtract a permutation matrix.

This time, we can actually do better: We can subtract the permutation matrix $\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$ from $\begin{pmatrix} 2 & 7 & \\ & & 9 \\ 7 & 2 & \end{pmatrix}$ not just once, but 7 times, without rendering any entry negative, because the relevant entries $7, 9, 7$ are all ≥ 7. The result is

$$\begin{pmatrix} 2 & 7 & \\ & & 9 \\ 7 & 2 & \end{pmatrix} - 7 \cdot \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} = \begin{pmatrix} 2 & & \\ & & 2 \\ & 2 & \end{pmatrix}.$$

Now, we follow the same recipe and again subtract a permutation matrix. This time, we can do it 2 times, and obtain

$$\begin{pmatrix} 2 & & \\ & & 2 \\ & 2 & \end{pmatrix} - 2 \cdot \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = 0_{3\times 3}$$

(the zero matrix, in case you're wondering).

Thus, we have arrived at the zero matrix by successively subtracting permutation matrices from M. Hence, M is the sum of all the permutation matrices that have been subtracted: namely,

$$M = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} + 7 \cdot \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix},$$

which is a sum of $1 + 7 + 2$ permutation matrices.

This method works in general, because:

- If M is an $\mathbb{N}$-magic matrix that is not the zero matrix, then Lemma 1.5.9 tells us that there is a permutation σ of $\{1, 2, \ldots, n\}$ such that all entries $M_{1,\sigma(1)}, M_{2,\sigma(2)}, \ldots, M_{n,\sigma(n)}$ are nonzero. We can then choose such a σ and subtract the corresponding permutation matrix $P(\sigma)$ from M.

- Better yet, we can subtract $m \cdot P(\sigma)$ from M, where

$$m = \min \left\{ M_{1,\sigma(1)}, M_{2,\sigma(2)}, \ldots, M_{n,\sigma(n)} \right\}.$$

This results in an $\mathbb{N}$-magic matrix (since the sum of the entries decreases by m in each row and by m in each column, and since we are only subtracting m from a bunch of entries that are $\geq m$) that has at least one fewer nonzero entry than M (since at least one of the nonzero entries $M_{1,\sigma(1)}, M_{2,\sigma(2)}, \ldots, M_{n,\sigma(n)}$ becomes 0 when m is subtracted from it).

- This way, in each step of our process, the number of nonzero entries of our matrix decreases by at least 1 (but the matrix remains an $\mathbb{N}$-magic matrix throughout the process). Hence, we eventually (after at most n^2 steps) will end up with the zero matrix.

This proves Theorem 1.5.7 **(a)**.

(b) This is analogous to the proof of part **(a)** (but this time, we **have** to subtract $m \cdot P(\sigma)$ rather than $P(\sigma)$ in our procedure, since the nonzero entries $M_{1,\sigma(1)}, M_{2,\sigma(2)}, \ldots, M_{n,\sigma(n)}$ are not necessarily ≥ 1).

(c) Let M be a doubly stochastic $n \times n$-matrix. Then, M is also $\mathbb{R}_+$-magic. Hence, part **(b)** shows that M can be expressed in the form $\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k$, where $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}_+$ are numbers and where $P_1, P_2, \ldots, P_k$ are permutation matrices. Consider these $\lambda_1, \lambda_2, \ldots, \lambda_k$ and these $P_1, P_2, \ldots, P_k$.

Now, consider the sum of all entries in the first row of M. It is easy to see that this sum is $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ (because $M = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k$, but each permutation matrix P_i contributes a 1 to the sum of all entries in the first row). But we know that this sum is 1, since M is doubly stochastic. Comparing these, we conclude that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$. Thus, we have expressed M in the form $\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k$, where $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}_+$ are numbers satisfying $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$ and where $P_1, P_2, \ldots, P_k$ are permutation matrices. This proves Theorem 1.5.7 **(c)**. □

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