# Math 530 Spring 2022, Lecture 24: Matchings

**website:** `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

# 1. Matchings

## 1.1. Introduction

Independent sets of a graph consist of vertices that "have no edges in common" (i.e., no two belong to the same edge).

In a sense, **matchings** are the dual notion to this: they consist of edges that "have no vertices in common" (i.e., no two contain the same vertex). Here is the formal definition:

**Definition 1.1.1.** Let $G = (V, E, \varphi)$ be a loopless multigraph.

**(a)** A **matching** of $G$ means a subset $M$ of $E$ such that no two distinct edges in $M$ have a common endpoint.

**(b)** If $M$ is a matching of $G$, then an $M$**-edge** shall mean an edge that belongs to $M$.

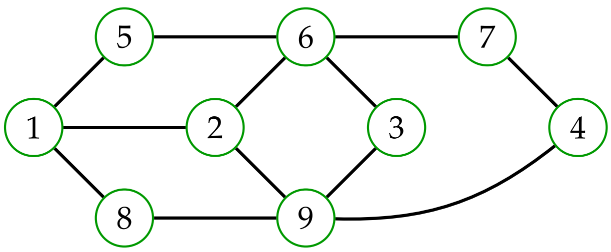
**(c)** If $M$ is a matching of $G$, and if $v \in V$ is any vertex, then we say that $v$ is **matched** in $M$ (or **saturated** in $M$) if $v$ is an endpoint of an $M$-edge. In this case, this latter $M$-edge is necessarily unique (since $M$ is a matching), and is called the $M$**-edge** of $v$. The other endpoint of this $M$-edge (i.e., its endpoint different from $v$) is called the $M$**-partner** of $v$.

**(d)** A matching $M$ of $G$ is said to be **perfect** if each vertex of $G$ is matched in $M$.

**(e)** Let $A$ be a subset of $V$. A matching $M$ of $G$ is said to be $A$**-complete** if each vertex in $A$ is matched in $M$.

Thus, a matching $M$ of a multigraph $G = (V, E, \varphi)$ is perfect if and only if it is $V$-complete.

**Exercise 1.** Let $G$ be the following simple graph:

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Then:

- The set $\{12,\ 36,\ 47\}$ is a matching of $G$. If we call this set $M$, then the vertices matched in $M$ are $1, 2, 3, 4, 6, 7$, and their respective $M$-partners are $2, 1, 6, 7, 3, 4$. This matching is not perfect, but it is (for example) $\{1, 3, 4\}$-complete and $\{1, 2, 3, 4, 6, 7\}$-complete.
- The set $\{12,\ 36,\ 67\}$ is not a matching of $G$, since the two distinct edges 36 and 67 from this set have a common endpoint.
- The sets $\varnothing$, $\{36\}$, $\{15,\ 29,\ 36,\ 47\}$ are matchings of $G$ as well.

We see that any matching "pairs up" some vertices using the existing edges of the graph. Clearly, the $M$-partner of the $M$-partner of a vertex $v$ is $v$ itself. Also, no two distinct vertices have the same $M$-partner (since otherwise, their $M$-edges would have a common endpoint).

**Remark 1.1.2.** A matching of a loopless multigraph $G = (V, E, \varphi)$ can also be characterized as a subset $M$ of its edge set $E$ such that all vertices of the spanning subgraph $(V, M, \varphi\mid_M)$ have degree $\leq 1$.

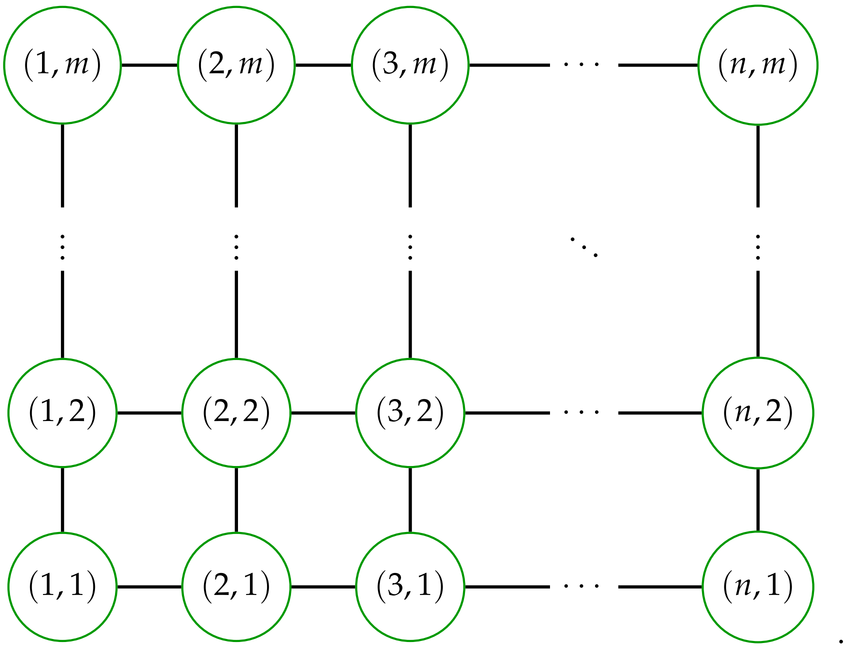
**Warning 1.1.3.** If a multigraph $G$ has loops, then most authors additionally require that a matching must not contain any loops. This ensures that Remark 1.1.2 remains valid.

Here are some natural questions:

- Does a given graph $G$ have a perfect matching?
- If not, can we find a maximum-size matching?
- What about an $A$-complete matching for a given $A \subseteq V$ ?

Some examples:

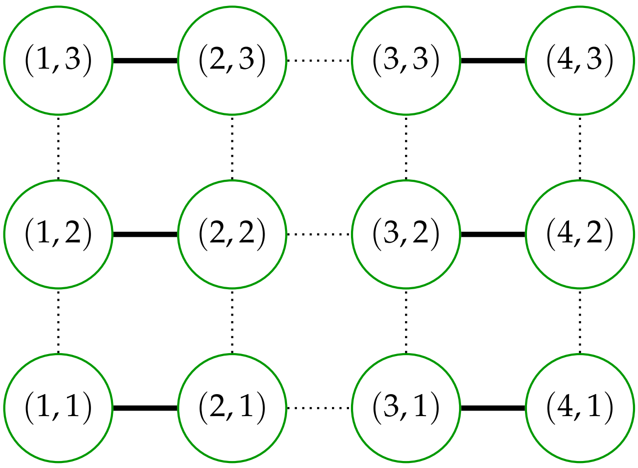
**Example 1.1.4.** Let $n$ and $m$ be two positive integers. The Cartesian product $P_n \times P_m$ of the $n$-th path graph $P_n$ and the $m$-th path graph $P_m$ is known as the $(n, m)$**-grid graph**, as it looks as follows:




**(a)** If $n$ is even, then

$$\{\{(i,j),\ (i+1,j)\} \mid i \text{ is odd, while } j \text{ is arbitrary}\}$$

is a perfect matching of $P_n \times P_m$. For example, here is this perfect matching for $n = 4$ and $m = 3$ (we have drawn all edges that do **not** belong to this matching as dotted lines):



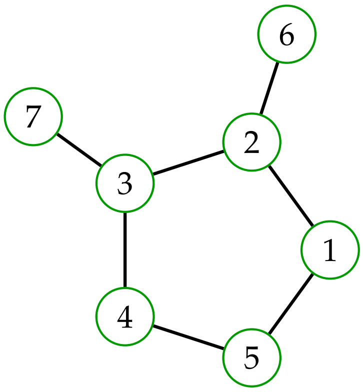

**(b)** Likewise, if $m$ is even, then

$$\{\{(i,j),\ (i,j+1)\} \mid j \text{ is odd, while } i \text{ is arbitrary}\}$$

is a perfect matching of $P_n \times P_m$.

**(c)** If $n$ and $m$ are both odd, then $P_n \times P_m$ has no perfect matching. Indeed, any loopless multigraph $G$ with an odd number of vertices cannot have a perfect matching, since each edge of the matching covers exactly 2 vertices.

**Example 1.1.5.** The "pentagon with two antlers" $C_5''$ (this is my notation, hopefully sufficiently natural) is the following graph:

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It has no perfect matching. This is easiest to see as follows: The graph $C_5''$ is loopless, so each edge contains exactly two vertices. Thus, any matching $M$ of $C_5''$ matches exactly $2 \cdot |M|$ vertices. In particular, any matching of $C_5''$ matches an even number of vertices. Since the total number of vertices $C_5''$ is odd, this entails that $C_5''$ has no perfect matching.

What is the maximum size of a matching of $C_5''$ ? The matching $\{12, 34\}$ of $C_5''$ has size 2 and cannot be improved by adding any new edges. Thus, one is tempted to believe that the maximum size of a matching is 2. However, this is not the case. Indeed, the matching $\{12, 37, 45\}$ has size 3. This latter matching is actually maximum-size.

Example 1.1.5 shows that when searching for a maximum-size matching, it is not sufficient to just keep adding edges until no further edges can be added; this strategy may lead to a non-improvable but non-maximum matching. This suggests that finding a maximum-size matching may be one of those hard problems like finding a maximum-size independent set. But no – there is a polynomial-time algorithm! It's known as the Edmonds blossom algorithm, and it has a running time of $O\left(|E| \cdot |V|^2\right)$; however, it is too complicated to be covered in this course. We shall here focus on a simple case of the problem that is already interesting enough and almost as useful as the general case.

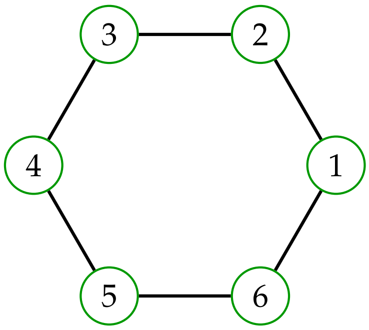
Namely, we shall study matchings of **bipartite graphs**.

## 1.2. Bipartite graphs

**Definition 1.2.1.** A **bipartite graph** means a triple $(G, X, Y)$, where

- $G = (V, E, \varphi)$ is a multigraph, and
- $X$ and $Y$ are two disjoint subsets of $V$ such that $X \cup Y = V$ and such that each edge of $G$ has one endpoint in $X$ and one endpoint in $Y$.

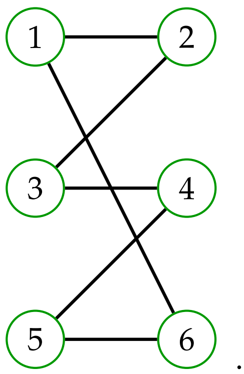
**Example 1.2.2.** Consider the 6-th cycle graph $C_6$:



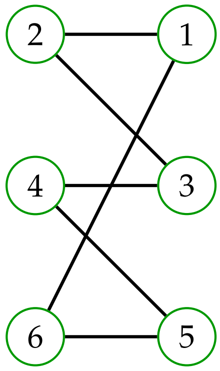

Then, $(C_6, \{1,3,5\}, \{2,4,6\})$ is a bipartite graph, since each edge of $G$ has one endpoint in $\{1,3,5\}$ and one endpoint in $\{2,4,6\}$. Also, $(C_6, \{2,4,6\}, \{1,3,5\})$ is a bipartite graph.

Note that a bipartite graph $(G, X, Y)$ is not just the graph $G$ but rather the whole package consisting of the graph $G$ and the subsets $X$ and $Y$. Two different bipartite graphs can have the same underlying graph $G$ but different choices of $X$ and $Y$. For example, the two bipartite graphs $(C_6, \{1,3,5\}, \{2,4,6\})$ and $(C_6, \{2,4,6\}, \{1,3,5\})$ are different.

We typically draw a bipartite graph $(G, X, Y)$ by drawing the graph $G$ in such a way that the vertices in $X$ are aligned along one vertical line and the vertices $Y$ are aligned along another, with the former line being left of the latter. Thus, for example, the bipartite graph $(C_6, \{1,3,5\}, \{2,4,6\})$ can be drawn as follows:



Similarly, the bipartite graph $(C_6, \{2,4,6\}, \{1,3,5\})$ can be drawn as follows:



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This example suggests the following terminology:

**Definition 1.2.3.** Let $(G, X, Y)$ be a bipartite graph. We shall refer to the vertices in $X$ as the **left vertices** of this bipartite graph. We shall refer to the vertices in $Y$ as the **right vertices** of this bipartite graph. Moreover, the edges of $G$ will be called the **edges** of this bipartite graph.

Thus, each edge of a bipartite graph joins one left vertex with one right vertex.

Bipartite graphs are "the same as" multigraphs with a proper 2-coloring. To wit:

**Proposition 1.2.4.** Let $G = (V, E, \varphi)$ be a multigraph.

**(a)** If $(G, X, Y)$ is a bipartite graph, then the map

$$f : V \to \{1,2\},$$
$$v \mapsto \begin{cases} 1, & \text{if } v \in X; \\ 2, & \text{if } v \in Y \end{cases}$$

is a proper 2-coloring of $G$.

**(b)** Conversely, if $f : V \to \{1,2\}$ is a proper 2-coloring of $G$, then $(G, V_1, V_2)$ is a bipartite graph, where we set

$$V_i := \{\text{all vertices with color } i\} \qquad \text{for each } i \in \{1,2\}.$$

**(c)** These constructions are mutually inverse. (That is, going from a bipartite graph to a proper 2-coloring and back again results in the original bipartite graph, whereas going from a proper 2-coloring to a bipartite graph and back again results in the original 2-coloring.)

*Proof.* An exercise in understanding the definitions. □

**Proposition 1.2.5.** Let $(G, X, Y)$ be a bipartite graph. Then, the graph $G$ has no circuits of odd length. In particular, $G$ has no loops or triangles.

*Proof.* By Proposition 1.2.4 **(a)**, we know that $G$ has a proper 2-coloring. Hence, the 2-coloring equivalence theorem (Theorem 1.1.1 in Lecture 22) shows that $G$ has no circuits of odd length. In particular, $G$ has no loops or triangles (since these would yield circuits of length 1 or 3, respectively). □

We need another piece of notation:

**Definition 1.2.6.** Let $G = (V, E, \varphi)$ be any multigraph. Let $U$ be a subset of $V$. Then,

$$N(U) := \{v \in V \mid v \text{ has a neighbor in } U\}.$$

This is called the **neighbor set** of $U$.

**Example 1.2.7.** If $G$ is the "pentagon with antlers" $C_5''$ from Example 1.1.5, then

$$N(\{1,5,6\}) = \{1,2,4,5\};$$
$$N(\{1\}) = \{2,5\};$$
$$N(\varnothing) = \varnothing.$$

For bipartite graphs, the neighbor set has a nice property:

**Proposition 1.2.8.** Let $(G, X, Y)$ be a bipartite graph. Let $A \subseteq X$. Then,

$$N(A) \subseteq Y.$$

*Proof.* Let $v \in N(A)$. Thus, the vertex $v$ has a neighbor in $A$ (by definition of $N(A)$). Let $w$ be this neighbor. Then, $w \in A \subseteq X$, so that $w \notin Y$ (since the bipartiteness of $(G, X, Y)$ shows that the sets $X$ and $Y$ are disjoint).

There exists some edge that has endpoints $v$ and $w$ (since $w$ is a neighbor of $v$). This edge must have an endpoint in $Y$ (since the bipartiteness of $(G, X, Y)$ shows that each edge of $G$ has one endpoint in $Y$). In other words, one of $v$ and $w$ must belong to $Y$ (since the endpoints of this edge are $v$ and $w$). Since $w \notin Y$, we thus conclude that $v \in Y$.

Thus, we have shown that $v \in Y$ for each $v \in N(A)$. In other words, $N(A) \subseteq Y$. □

## 1.3. Hall's marriage theorem

How can we tell whether a bipartite graph has a perfect matching? an $X$-complete matching? First, to keep the suspense, let us prove some trivialities:

**Proposition 1.3.1.** Let $(G, X, Y)$ be a bipartite graph. Let $M$ be a matching of $G$. Then:

- **(a)** The $M$-partner of a vertex $x \in X$ (if it exists) belongs to $Y$.

  The $M$-partner of a vertex $y \in Y$ (if it exists) belongs to $X$.

- **(b)** We have $|M| \leq |X|$ and $|M| \leq |Y|$.

- **(c)** If $M$ is $X$-complete, then $|X| \leq |Y|$.

- **(d)** If $M$ is perfect, then $|X| = |Y|$.

- **(e)** If $|M| \geq |X|$, then $M$ is $X$-complete.

- **(f)** If $M$ is $X$-complete and we have $|X| = |Y|$, then $M$ is perfect.

*Proof.* Each edge of $G$ has an endpoint in $X$ and an endpoint in $Y$ (since $(G, X, Y)$ is a bipartite graph). Thus, in particular, each $M$-edge has an endpoint in $X$ and an endpoint in $Y$. Moreover, no two $M$-edges share a common endpoint (since $M$ is a matching).

**(a)** This follows from the fact that each $M$-edge has an endpoint in $X$ and an endpoint in $Y$.

**(b)** Recall that each $M$-edge has an endpoint in $X$. Since no two $M$-edges share a common endpoint, we thus have found at least $|M|$ many endpoints in $X$. This entails $|M| \leq |X|$. Similarly, $|M| \leq |Y|$.

**(c)** Assume that $M$ is $X$-complete. Hence, each vertex in $X$ is matched in $M$ and therefore has an $M$-edge that contains it. In other words, for each vertex $x \in X$, there exists an $M$-edge $m$ such that $x$ is an endpoint of $m$. Since no two $M$-edges share an endpoint, this yields that there are at least $|X|$ many $M$-edges. In other words, $|M| \geq |X|$. Hence, $|X| \leq |M| \leq |Y|$ (by part **(b)**).

**(d)** Assume that $M$ is perfect. Then, $M$ is both $X$-complete and $Y$-complete. Hence, part **(c)** yields $|X| \leq |Y|$; similarly, $|Y| \leq |X|$. Combining these two inequalities, we obtain $|X| = |Y|$.

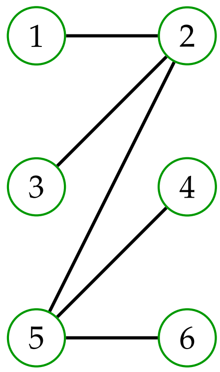
**(e)** Assume that $|M| \geq |X|$.

However, each $M$-edge has an endpoint in $X$. These endpoints are all distinct (since no two $M$-edges share a common endpoint), and there are at least $|X|$ many of them (since there are $|M|$ many of them, but we have $|M| \geq |X|$). Therefore, these endpoints must cover **all** the vertices in $X$ (because the only way to choose $|X|$ many distinct vertices in $X$ is to choose **all** vertices in $X$). In other words, all the vertices in $X$ must be matched in $M$. In other words, the matching $M$ is $X$-complete.

**(f)** Assume that $M$ is $X$-complete and that we have $|X| = |Y|$.

The matching $M$ is $X$-complete; thus, all vertices $x \in X$ are matched in $M$. The $M$-partners of all these vertices $x \in X$ belong to $Y$ (by Proposition 1.3.1 **(a)**), and are also matched in $M$. Hence, at least $|X|$ many vertices in $Y$ must be matched in $M$ (since these $M$-partners are all distinct[1]). In other words, at least $|Y|$ many vertices in $Y$ must be matched in $M$ (since $|X| = |Y|$). This means that **all** vertices in $Y$ are matched in $M$ (since "at least $|Y|$ many vertices in $Y$" means "all vertices in $Y$"). Since we also know that all vertices $x \in X$ are matched in $M$, we thus conclude that all vertices of $G$ are matched in $M$. In other words, the matching $M$ is perfect. $\square$

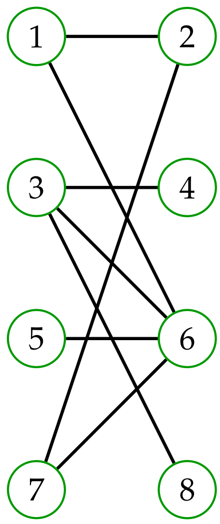
**Example 1.3.2.** Consider the bipartite graph




(drawn as explained in Example 1.2.2). Does this graph have a perfect matching? No, because the two left vertices 1 and 3 would necessarily have the same partner in such a matching (since their only possible partner is 2).

[1] because the $M$-partners of distinct vertices are distinct

Similarly, the bipartite graph




has no perfect matching, since the three left vertices 1, 5 and 7 have only two potential partners (viz., 2 and 6).

So we see that a subset $A \subseteq X$ satisfying $|N(A)| < |A|$ is an obstruction to the existence of an $X$-complete matching. Let us state this in a positive way:

**Proposition 1.3.3.** Let $(G, X, Y)$ be a bipartite graph. Let $A$ be a subset of $X$. Assume that $G$ has an $X$-complete matching. Then, $|N(A)| \geq |A|$.

*Proof.* Let $V$ be the vertex set of $G$. We assumed that $G$ has an $X$-complete matching. Let $M$ be such a matching. Thus, each $x \in X$ has an $M$-partner. The map

$$\begin{aligned} \mathbf{p} : X &\to V, \\ x &\mapsto (\text{the } M\text{-partner of } x) \end{aligned}$$

is injective (since two distinct vertices cannot have the same $M$-partner). Thus, $|\mathbf{p}(A)| = |A|$ (because any injective map preserves the size of a subset). However, $\mathbf{p}(A) \subseteq N(A)$, because the $M$-partner of an element of $A$ will always belong to $N(A)$. Hence, $|\mathbf{p}(A)| \leq |N(A)|$. Thus, $|N(A)| \geq |\mathbf{p}(A)| = |A|$, qed. $\square$

So we have found a necessary condition for the existence of an $X$-complete matching. Interestingly, it is also sufficient:

**Theorem 1.3.4** (Hall's marriage theorem, short: HMT)**.** Let $(G, X, Y)$ be a bipartite graph. Assume that each subset $A$ of $X$ satisfies $|N(A)| \geq |A|$. (This assumption is called the "**Hall condition**".)

Then, $G$ has an $X$-complete matching.

This is called "marriage theorem" because one can interpret a bipartite graph as a dating scene, with $X$ being the guys and $Y$ the ladies. A guy $x$ and a lady $y$ are adjacent if and only if they are interested in one another. Thus, an $X$-complete matching is a way of marrying off each guy to some lady he is mutually interested in (without allowing polygamy). This is a classical model for bipartite graphs and appears all across the combinatorics literature; to my knowledge, however, no real-life applications have been found along these lines. Nevertheless, Hall's marriage theorem can be applied in many other situations, such as logistics (although its generalizations, which we will soon see, are even more useful in that). Philip Hall has originally invented the theorem in 1935, motivated (I believe) by a problem about finite groups. So did Wilhelm Maak, also in 1935, for use in analysis (defining a notion of integrals for almost-periodic functions).

There are many proofs of Hall's marriage theorem, some pretty easy. Two short and self-contained proofs can be found in [LeLeMe17, §12.5.2] and in [Harju14, Theorem 3.9]. I will tease you by leaving the theorem unproved for this and the next class, while exploring some of its many consequences. Afterwards (in Lecture 26), I will give a proof using the theory of **network flows** – an elegant theory created for use in logistics[2] in the 1950s that has proved to be quite useful in combinatorics. Among other consequences, this proof will also provide a polynomial-time algorithm for actually finding a maximum matching in a bipartite graph (Theorem 1.3.4 by itself does not help here).

## 1.4. König and Hall-König

Hall's marriage theorem is famous for its many forms and versions, most of which are "secretly" equivalent to it (i.e., can be derived from it and conversely can be used to derive it without too much trouble). We will start with one that is known as **König's theorem** (discovered independently by Dénes Kőnig and Jenő Egerváry in 1931). This relies on the notion of a **vertex cover**. Here is its definition:

**Definition 1.4.1.** Let $G = (V, E, \varphi)$ be a multigraph. A **vertex cover** of $G$ means a subset $C$ of $V$ such that each edge of $G$ contains at least one vertex in $C$.

**Example 1.4.2.** Let $n \geq 1$. What are the vertex covers of the complete graph $K_n$ ?

A quick thought reveals that any subset $S$ of $\{1, 2, \ldots, n\}$ that has at least $n-1$ elements is a vertex cover of $K_n$. (In fact, $K_n$ has no loops, so that each edge of $K_n$ contains two different vertices, and thus at least one of these two vertices belongs to $S$.) On the other hand, a subset $S$ with fewer than $n-1$ vertices will never be a vertex cover of $K_n$ (since there will be at least

[2] and, more generally, operations research

two distinct vertices that don't belong to $S$, and the edge that joins these two vertices contains no vertex in $S$).

**Example 1.4.3.** Let $G = (V, E, \varphi)$ be the graph from Example 1.3.2. Then, the set $\{2, 5\}$ is a vertex cover of $G$. Of course, any subset of $V$ that contains $\{2, 5\}$ as a subset will thus also be a vertex cover of $G$.

Note that the notion of a vertex cover is (in some sense) "dual" to the notion of an edge cover, which we defined on homework set #1. For those getting confused, here is a convenient table (courtesy of Nadia Lafrenière, Math 38, Spring 2021):

| **a ...** | **is a set of ...** | **that contains ...** |
|---|---|---|
| matching | edges | at most one edge per vertex |
| edge cover | edges | at least one edge per vertex |
| independent set | vertices | at most one vertex per edge |
| vertex cover | vertices | at least one vertex per edge |

The notion of vertex covers is also somewhat reminiscent of the notion of dominating sets; here is the precise relation:

**Remark 1.4.4.** Each vertex cover of a multigraph $G$ is a dominating set (as long as $G$ has no vertices of degree 0). But the converse is not true.

**Proposition 1.4.5.** Let $G$ be a loopless multigraph.
Let $m$ be the largest size of a matching of $G$.
Let $c$ be the smallest size of a vertex cover of $G$.
Then, $m \leq c$.

*Proof.* By the definition of $m$, we know that $G$ has a matching $M$ of size $m$.

By the definition of $c$, we know that $G$ has a vertex cover $C$ of size $c$.

Consider these $M$ and $C$. Every $M$-edge $e \in M$ contains at least one vertex in $C$ (since $C$ is a vertex cover). Thus, we can define a map $f : M \to C$ that sends each $M$-edge $e$ to some vertex in $C$ that is contained in $e$. (If there are two such vertices, then we just pick one of them at random.) This map $f$ is injective, because no two $M$-edges contain the same vertex (after all, $M$ is a matching). Thus, we have found an injective map from $M$ to $C$ (namely, $f$). Therefore, $|M| \leq |C|$. But the definitions of $M$ and $C$ show that $|M| = m$ and $|C| = c$. Thus, $m = |M| \leq |C| = c$, and Proposition 1.4.5 is proved. $\square$

In general, we can have $m < c$ in Proposition 1.4.5. However, for a **bipartite** graph, equality reigns:

**Theorem 1.4.6** (König's theorem)**.** Let $(G, X, Y)$ be a bipartite graph.
Let $m$ be the largest size of a matching of $G$.
Let $c$ be the smallest size of a vertex cover of $G$.
Then, $m = c$.

Both Hall's and König's theorems easily follow from the following theorem:

**Theorem 1.4.7** (Hall-König matching theorem)**.** Let $(G, X, Y)$ be a bipartite graph. Then, there exist a matching $M$ of $G$ and a subset $U$ of $X$ such that

$$|M| \geq |N(U)| + |X| - |U|.$$

We will prove this theorem in Lecture 26 and again in Lecture 28. For now, let us show that Hall's marriage theorem (Theorem 1.3.4), König's theorem (Theorem 1.4.6) and the Hall-König matching theorem (Theorem 1.4.7) are mutually equivalent. More precisely, we will explain how to derive the first two from the third, and outline the reverse derivations.

*Proof of Theorem 1.3.4 using Theorem 1.4.7.* Assume that Theorem 1.4.7 has already been proved.

Theorem 1.4.7 yields that there exist a matching $M$ of $G$ and a subset $U$ of $X$ such that

$$|M| \geq |N(U)| + |X| - |U|.$$

Consider these $M$ and $U$. The Hall condition shows that each subset $A$ of $X$ satisfies $|N(A)| \geq |A|$. Applying this to $A = U$, we obtain $|N(U)| \geq |U|$. Thus,

$$|M| \geq \underbrace{|N(U)|}_{\geq |U|} + |X| - |U| \geq |X|.$$

Hence, the matching $M$ is $X$-complete (by Proposition 1.3.1 **(e)**). Thus, we have found an $X$-complete matching. This proves Theorem 1.3.4 (assuming that Theorem 1.4.7 is true). □

*Proof of Theorem 1.4.6 using Theorem 1.4.7.* Assume that Theorem 1.4.7 has already been proved.

Write the multigraph $G$ as $G = (V, E, \varphi)$. Theorem 1.4.7 yields that there exist a matching $M$ of $G$ and a subset $U$ of $X$ such that

$$|M| \geq |N(U)| + |X| - |U|. \tag{1}$$

Consider these $M$ and $U$. Clearly, $|M| \leq m$ (since $m$ is the largest size of a matching of $G$).

Let $C := (X \setminus U) \cup N(U)$. This is a subset of $V$. Moreover, each edge of $G$ has at least one endpoint in $C$ (this is easy to see[3]). Hence, $C$ is a vertex cover

[3] *Proof.* Let $e$ be an edge of $G$. We must show that $e$ has at least one endpoint in $C$.

of $G$. Therefore, $|C| \geq c$ (since $c$ is the smallest size of a vertex cover of $G$). The definition of $C$ yields

$$\begin{aligned}
|C| &= |(X \setminus U) \cup N(U)| \\
&\leq \underbrace{|X \setminus U|}_{\substack{=|X|-|U| \\ (\text{since } U \subseteq X)}} + |N(U)| \qquad \text{(actually an equality, but we don't care)} \\
&= |X| - |U| + |N(U)| = |N(U)| + |X| - |U| \leq |M| \qquad \text{(by (1))} \\
&\leq m.
\end{aligned}$$

Hence, $m \geq |C| \geq c$. Combining this with $m \leq c$ (which follows from Proposition 1.4.5), we obtain $m = c$. Thus, Theorem 1.4.6 follows. □

Conversely, it is not hard to derive the HKMT from either Hall or König:

*Proof of Theorem 1.4.7 using Theorem 1.3.4 (sketched).* Assume that Theorem 1.3.4 has already been proved.

Add a bunch of "dummy vertices" to $Y$ and join each of these "dummy vertices" by a new edge to each vertex in $X$. How many "dummy vertices" should we add? As many as it takes to ensure that every subset $A$ of $X$ satisfies the Hall condition – i.e., exactly $\max\{|A| - |N(A)| \mid A \text{ is a subset of } X\}$ many.

Let $G'$ be the resulting graph. Let also $D$ be the set of all dummy vertices that were added to $Y$, and let $Y' = Y \cup D$ be the set of all right vertices of $G'$. (The set of left vertices of $G'$ is still $X$.) Then, the bipartite graph $(G', X, Y')$ satisfies the Hall condition, and therefore we can apply Theorem 1.3.4 to $(G', X, Y')$ instead of $(G, X, Y)$, and conclude that the graph $G'$ has an $X$-complete matching. Let $M'$ be this matching. By removing from $M'$ all edges that contain dummy vertices, we obtain a matching $M$ of $G$. This matching $M$ has size

$$\begin{aligned}
|M| &= |M'| - \underbrace{(\text{the number of edges that were removed from } M')}_{\substack{\leq (\text{the number of dummy vertices}) \\ (\text{since each dummy vertex is contained in at most one } M'\text{-edge})}} \\
&\geq |M'| - \underbrace{(\text{the number of dummy vertices})}_{\substack{=\max\{|A|-|N(A)| \mid A \text{ is a subset of } X\} \\ (\text{by the construction of the dummy vertices})}} \\
&= |M'| - \max\{|A| - |N(A)| \mid A \text{ is a subset of } X\}. \qquad (2)
\end{aligned}$$

---

Clearly, the edge $e$ has an endpoint in $X$ (since $(G, X, Y)$ is a bipartite graph). Let $x$ be this endpoint. This $x$ either belongs to $U$ or doesn't.

- If $x$ belongs to $U$, then the other endpoint of $e$ (that is, the endpoint distinct from $x$) belongs to $N(U)$ (since its neighbor $x$ belongs to $U$) and therefore to $C$ (since $N(U) \subseteq (X \setminus U) \cup N(U) = C$).
- If $x$ does not belong to $U$, then $x$ belongs to $X \setminus U$ (since $x \in X$) and therefore to $C$ (since $X \setminus U \subseteq (X \setminus U) \cup N(U) = C$).

In either of these two cases, we have found an endpoint of $e$ that belongs to $C$. Thus, $e$ has at least one endpoint in $C$, qed.

However, the maximum of a set is always an element of this set. Hence, there exists a subset $U$ of $X$ such that

$$\max \{|A| - |N(A)| \mid A \text{ is a subset of } X\} = |U| - |N(U)|.$$

Consider this $U$. Then, (2) becomes

$$\begin{aligned} |M| &\geq \underbrace{|M'|}_{\substack{\geq |X| \\ \text{(since } M' \text{ is } X\text{-complete,} \\ \text{and thus each } x \in X \text{ has} \\ \text{an } M'\text{-edge (and these} \\ \text{edges are distinct))}}} - \underbrace{\max \{|A| - |N(A)| \mid A \text{ is a subset of } X\}}_{= |U| - |N(U)|} \\ &\geq |X| - (|U| - |N(U)|) = |N(U)| + |X| - |U|. \end{aligned}$$

Hence, we have found a matching $M$ of $G$ and a subset $U$ of $X$ such that $|M| \geq |N(U)| + |X| - |U|$. This proves Theorem 1.4.7 (assuming that Theorem 1.3.4 is true). □

*Proof of Theorem 1.4.7 from Theorem 1.4.6 (sketched).* Assume that Theorem 1.4.6 has already been proved.

Let $M$ be a maximum-size matching of $G$. Let $C$ be a minimum-size vertex cover of $G$. Then, Theorem 1.4.6 says that $|M| = |C|$.

Let $U := X \setminus C$. Then, $N(U) \subseteq C \setminus X$ (why?). Hence, $|N(U)| \leq |C \setminus X|$, so that

$$\underbrace{|N(U)|}_{\leq |C \setminus X|} + |X| - \left| \underbrace{U}_{= X \setminus C} \right| \leq |C \setminus X| + \underbrace{|X| - |X \setminus C|}_{= |C \cap X|} = |C \setminus X| + |C \cap X| = |C| = |M|.$$

Hence, $|M| \geq |N(U)| + |X| - |U|$. This proves Theorem 1.4.7 (assuming that Theorem 1.4.6 is true). □

Theorem 1.4.7 thus occupies a convenient "high ground" between the Hall and König theorems, allowing easy access to both of them. We shall prove Theorem 1.4.7 in Lecture 26.

# References

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