Math 530 Spring 2022, Lecture 23: Independent sets

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Independent sets

1.1. Definition and lower bound

Next, we define one of the most fundamental notions in graph theory:

Definition 1.1.1. An **independent set** of a multigraph G means a subset S of $\operatorname{V}(G)$ such that no two elements of S are adjacent.

In other words, an independent set of G means an induced subgraph of G that has no edges[1]. Note that "no two elements of S" doesn't mean "no two distinct elements of S".

Thus, for example, what we called an "anti-triangle" (back in Lecture 1) is an independent set of size 3.

Remark 1.1.2. Independent sets are closely related to proper colorings. Indeed, let G be a graph, and let $k \in \mathbb{N}$. Let $f : V \to \{1, 2, \ldots, k\}$ be a k-coloring. For each $i \in \{1, 2, \ldots, k\}$, let

$$\begin{aligned} V_i &:= \{v \in V \mid f(v) = i\} \\ &= \{\text{all vertices of } G \text{ that have color } i\}. \end{aligned}$$

Then, the k-coloring f is proper if and only if the k sets $V_1, V_2, \ldots, V_k$ are independent sets of G. (Proving this is a matter of unraveling the definitions of "independent sets" and "proper k-colorings".)

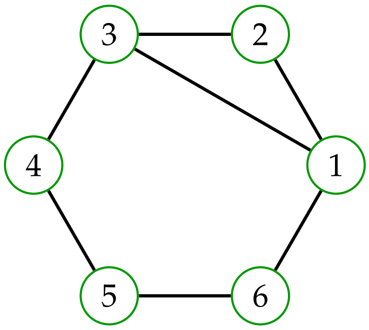
One classical computational problem in graph theory is to find a maximum-size independent set of a given graph. This problem is NP-hard, so don't expect a quick algorithm or even a good formula for the maximum size of an independent set. However, there are some lower bounds for this maximum size. Here is one:

Theorem 1.1.3. Let $G = (V, E, \varphi)$ be a loopless multigraph. Then, G has an independent set of size

$$\geq \sum_{v \in V} \frac{1}{1 + \deg v}.$$

[1] This is a somewhat sloppy statement. Of course, an independent set is not literally an induced subgraph, since the former is just a set, while the latter is a graph. What I mean is that a subset S of $\operatorname{V}(G)$ is independent if and only if the induced subgraph $G[S]$ has no edges.

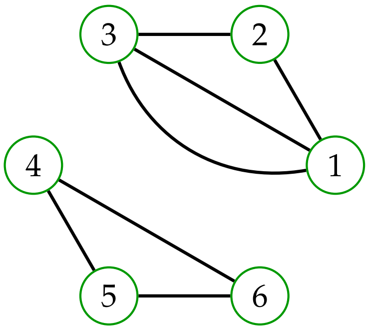
Example 1.1.4. Let G be the following loopless multigraph:



Then, the degrees of the vertices of G are $3, 2, 3, 2, 2, 2$. Hence, Theorem 1.1.3 yields that G has an independent set of size

$$\geq \frac{1}{1+3} + \frac{1}{1+2} + \frac{1}{1+3} + \frac{1}{1+2} + \frac{1}{1+2} + \frac{1}{1+2} = \frac{11}{6} \approx 1.83.$$

Since the size of an independent set is always an integer, we can round this up and conclude that G has an independent set of size ≥ 2. In truth, G actually has an independent set of size 3 (namely, $\{2, 4, 6\}$), but there is no way to tell this from the degrees of its vertices alone. For example, the vertices of the graph

$H :=$ 

have the same degrees as those of G, but H has no independent set of size 3.

We shall give two proofs of Theorem 1.1.3, both of them illustrating useful techniques.[2]

First proof of Theorem 1.1.3. Assume the contrary. Thus, each independent set S of G has size

$$|S| < \sum_{v \in V} \frac{1}{1 + \deg v}. \tag{1}$$

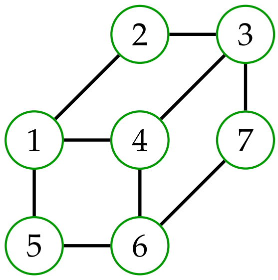
A V**-listing** shall mean a list of all vertices in V, with each vertex occurring exactly once in the list. If σ is a V-listing, then we define a subset J_σ of V as

[2]Note that the looplessness requirement in Theorem 1.1.3 is important: If G has a loop at each vertex, then the only independent set of G is $\varnothing$.

follows:

$$J_\sigma := \{v \in V \mid v \text{ occurs } \textbf{before} \text{ all neighbors of } v \text{ in } \sigma\}.$$

[**Example:** Let G be the following graph:



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Let σ be the V-listing $(1, 2, 7, 5, 3, 4, 6)$. Then, the vertex 1 occurs before all its neighbors (2, 4 and 5) in σ, and thus we have $1 \in J_\sigma$. Likewise, the vertex 7 occurs before all its neighbors (3 and 6) in σ, so that we have $7 \in J_\sigma$. But the vertex 2 does not occur before all its neighbors in σ (indeed, it occurs after its neighbor 1), so that we have $2 \notin J_\sigma$. Likewise, the vertices $5, 3, 4, 6$ don't belong to J_σ. Altogether, we thus obtain $J_\sigma = \{1, 7\}$.]

The set J_σ is an independent set of G (because if two vertices u and v in J_σ were adjacent, then u would have to occur before v in σ, but v would have to occur before u in σ; but these two statements clearly contradict each other). Thus, (1) (applied to $S = J_\sigma$) yields

$$|J_\sigma| < \sum_{v \in V} \frac{1}{1 + \deg v}.$$

This inequality holds for **each** V-listing σ. Thus, summing this inequality over all V-listings σ, we obtain

$$\begin{aligned} \sum_{\sigma \text{ is a } V\text{-listing}} |J_\sigma| &< \sum_{\sigma \text{ is a } V\text{-listing}} \sum_{v \in V} \frac{1}{1 + \deg v} \\ &= (\text{\# of all } V\text{-listings}) \cdot \sum_{v \in V} \frac{1}{1 + \deg v}. \end{aligned} \tag{2}$$

On the other hand, I claim the following:

Claim 1: For each $v \in V$, we have

$$(\text{\# of all } V\text{-listings } \sigma \text{ satisfying } v \in J_\sigma) \geq \frac{(\text{\# of all } V\text{-listings})}{1 + \deg v}.$$

[*Proof of Claim 1:* Fix a vertex $v \in V$. Define $\deg' v$ to be the # of all neighbors of v. Clearly, $\deg' v \leq \deg v$.

We shall call a V-listing σ **good** if the vertex v occurs in it before all its neighbors. In other words, a V-listing σ is good if and only if it satisfies $v \in J_\sigma$ (because $v \in J_\sigma$ means that the vertex v occurs in σ before all its neighbors[3]). Thus, we must show that

$$(\text{\# of all good } V\text{-listings}) \geq \frac{(\text{\# of all } V\text{-listings})}{1 + \deg v}.$$

We define a map

$$\Gamma : \{\text{all } V\text{-listings}\} \to \{\text{all good } V\text{-listings}\}$$

as follows: Whenever τ is a V-listing, we let $\Gamma(\tau)$ be the V-listing obtained from τ by swapping v with the first neighbor of v that occurs in τ (or, if τ is already good, then we just do nothing, i.e., we set $\Gamma(\tau) = \tau$). This map Γ is a $(1 + \deg' v)$-to-1 correspondence – i.e., for each good V-listing σ, there are exactly $1 + \deg' v$ many V-listings τ that satisfy $\Gamma(\tau) = \sigma$ (in fact, one of these τ's is σ itself, and the remaining $\deg' v$ many of these τ's are obtained from σ by switching v with some neighbor of v). Hence, by the multijection principle[4], we conclude that

$$|\{\text{all } V\text{-listings}\}| = (1 + \deg' v) \cdot |\{\text{all good } V\text{-listings}\}|.$$

In other words,

$$(\text{\# of all } V\text{-listings}) = (1 + \deg' v) \cdot (\text{\# of all good } V\text{-listings}).$$

Hence,

$$(\text{\# of all good } V\text{-listings}) = \frac{(\text{\# of all } V\text{-listings})}{1 + \deg' v} \geq \frac{(\text{\# of all } V\text{-listings})}{1 + \deg v}$$

(since $\deg' v \leq \deg v$). This proves Claim 1 (since the good V-listings are precisely the V-listings σ satisfying $v \in J_\sigma$).]

Next, we recall a basic property of the Iverson bracket notation[5]: If T is a subset of a finite set S, then

$$|T| = \sum_{v \in S} [v \in T]. \tag{3}$$

(Indeed, the sum $\sum\limits_{v \in S} [v \in T]$ contains an addend equal to 1 for each $v \in T$, and an addend equal to 0 for each $v \in S \setminus T$. Thus, this sum amounts to $|T| \cdot 1 + |S \setminus T| \cdot 0 = |T|$.)

[3] This follows straight from the definition of J_σ.

[4] See a footnote in Lecture 17 for the statement of the multijection principle.

[5] See, e.g., Lecture 18 for the definition of the Iverson bracket notation.

Now, (2) yields

$$\begin{aligned}
&(\text{\# of all } V\text{-listings}) \cdot \sum_{v \in V} \frac{1}{1+\deg v} \\
&> \sum_{\sigma \text{ is a } V\text{-listing}} \underbrace{|J_\sigma|}_{\substack{=\sum\limits_{v\in V}[v\in J_\sigma] \\ \text{(by (3))}}} = \underbrace{\sum_{\sigma \text{ is a } V\text{-listing}} \sum_{v\in V}}_{=\sum\limits_{v\in V} \sum\limits_{\sigma \text{ is a } V\text{-listing}}} [v \in J_\sigma] \\
&= \sum_{v\in V} \underbrace{\sum_{\sigma \text{ is a } V\text{-listing}} [v \in J_\sigma]}_{\substack{=(\text{\# of all } V\text{-listings } \sigma \text{ satisfying } v\in J_\sigma) \\ \text{(because the sum } \sum\limits_{\sigma \text{ is a } V\text{-listing}} [v\in J_\sigma] \\ \text{contains an addend equal to 1 for each } V\text{-listing } \sigma \text{ satisfying } v\in J_\sigma, \\ \text{and an addend equal to 0 for each other } V\text{-listing } \sigma)}} \\
&= \sum_{v\in V} \underbrace{(\text{\# of all } V\text{-listings } \sigma \text{ satisfying } v \in J_\sigma)}_{\substack{\geq \frac{(\text{\# of all } V\text{-listings})}{1+\deg v} \\ \text{(by Claim 1)}}} \\
&\geq \sum_{v\in V} \frac{(\text{\# of all } V\text{-listings})}{1+\deg v} = (\text{\# of all } V\text{-listings}) \cdot \sum_{v\in V} \frac{1}{1+\deg v}.
\end{aligned}$$

This is absurd (since no real number x can satisfy $x > x$). So we got a contradiction, and our proof of Theorem 1.1.3 is complete. □

Remark 1.1.5. This proof is an example of a **probabilistic proof**. Why? We have been manipulating sums, but we could easily replace these sums by averages. Claim 1 then would say the following: For any given vertex $v \in V$, the **probability** that a (uniformly random) V-listing σ satisfies $v \in J_\sigma$ is $\geq \frac{1}{1+\deg v}$. Thus, the expectation of $|J_\sigma|$ is $\geq \sum_{v\in V} \frac{1}{1+\deg v}$ (by linearity of expectation). Therefore, at least one V-listing σ actually satisfies $|J_\sigma| \geq \sum_{v\in V} \frac{1}{1+\deg v}$. So the whole proof can be restated in terms of probabilities and expectations.

Note that this proof (as it stands) is fairly useless as it comes to actually **finding** an independent set of size $\geq \sum_{v\in V} \frac{1}{1+\deg v}$. It does not give any better algorithm than "try the subsets J_σ for all possible V-listings σ; one of them will work", which is even slower than trying all subsets of V.

Note also that the proof does **not** entail that at least half of the V-listings σ will satisfy $|J_\sigma| \geq \sum_{v\in V} \frac{1}{1+\deg v}$. The mean is not the median!

Let us now give a second proof of the theorem, which does provide a good algorithm:

Second proof of Theorem 1.1.3. We proceed by strong induction on $|V|$. Thus, we fix $p \in \mathbb{N}$, and we assume (as the induction hypothesis) that Theorem 1.1.3 is already proved for all loopless multigraphs G with $< p$ vertices. We must now prove it for a loopless multigraph $G = (V, E, \varphi)$ with p vertices.

If $|V| = 0$, then this is clear (since $\varnothing$ is an independent set of appropriate size). Thus, we WLOG assume that $|V| \neq 0$. We furthermore assume WLOG that G is a simple graph (because otherwise, we can replace G by G^{simp}; this can only decrease the degrees $\deg v$ of the vertices $v \in V$, and thus our claim only becomes stronger).

Since $|V| \neq 0$, there exists a vertex $u \in V$ with $\deg_G u$ minimum[6]. Pick such a u. Thus,

$$\deg_G v \geq \deg_G u \qquad \text{for each } v \in V. \tag{4}$$

Let $U := \{u\} \cup \{\text{all neighbors of } u\}$. Thus, $U \subseteq V$ and $|U| = 1 + \deg_G u$ (this is a honest equality, since G is a simple graph).

Let G' be the induced subgraph of G on the set $V \setminus U$. This is the simple graph obtained from G by removing all vertices belonging to U (that is, removing the vertex u along with all its neighbors) and removing all edges that require these vertices. Then, G' has fewer vertices than G. Hence, G' has $< p$ vertices (since G has p vertices). Hence, by the induction hypothesis, Theorem 1.1.3 is already proved for G'. In other words, G' has an independent set of size

$$\geq \sum_{v \in V \setminus U} \frac{1}{1 + \deg_{G'} v}.$$

Let T be such an independent set. Set $S := \{u\} \cup T$. Then, S is an independent set of G (since $T \subseteq V \setminus U$, so that T contains no neighbors of u). Moreover, I claim that $|S| \geq \sum_{v \in V} \frac{1}{1 + \deg_G v}$. Indeed, this follows from

$$\begin{aligned}
\sum_{v \in V} \frac{1}{1 + \deg_G v} &= \sum_{v \in U} \underbrace{\frac{1}{1 + \deg_G v}}_{\substack{\leq \frac{1}{1 + \deg_G u} \\ \text{(since } \deg_G v \geq \deg_G u \\ \text{(by (4)))}}} + \sum_{v \in V \setminus U} \underbrace{\frac{1}{1 + \deg_G v}}_{\substack{\leq \frac{1}{1 + \deg_{G'} v} \\ \text{(since } \deg_G v \geq \deg_{G'} v \\ \text{(because } G' \text{ is a subgraph of } G\text{))}}} \\
&\leq \underbrace{\sum_{v \in U} \frac{1}{1 + \deg_G u}}_{\substack{= |U| \cdot \frac{1}{1 + \deg_G u} = 1 \\ \text{(since } |U| = 1 + \deg_G u\text{)}}} + \underbrace{\sum_{v \in V \setminus U} \frac{1}{1 + \deg_{G'} v}}_{\substack{\leq |T| \\ \text{(since } T \text{ has size } \geq \sum_{v \in V \setminus U} \frac{1}{1 + \deg_{G'} v}\text{)}}} \\
&\leq 1 + |T| = |S| \qquad (\text{since } S = \{u\} \cup T).
\end{aligned}$$

[6] Here, the notation $\deg_H u$ means the degree of a vertex u in a graph H.

So we have found an independent set of G having size $\geq \sum_{v \in V} \frac{1}{1 + \deg_G v}$ (namely, S). This means that Theorem 1.1.3 holds for our G. This completes the induction step, and Theorem 1.1.3 is proved. □

Remark 1.1.6. The second proof of Theorem 1.1.3 (unlike the first one) does give a fairly efficient algorithm for finding an independent set of the appropriate size. However, the second proof is actually not that much different from the first proof; it can in fact be recovered from the first proof by **derandomization**, specifically using the **method of conditional probabilities**. (This is a general technique for "derandomizing" probabilistic proofs, i.e., turning them into algorithmic ones. It often requires some ingenuity and is not guaranteed to always work, but the above is an example where it can be applied. See [Aspnes23, Chapter 13] for much more about derandomization.)

1.2. A weaker (but simpler) lower bound

Let us now weaken Theorem 1.1.3 a bit:

Corollary 1.2.1. Let G be a loopless multigraph with n vertices and m edges. Then, G has an independent set of size

$$\geq \frac{n^2}{n + 2m}.$$

In order to prove this, we will need the following inequality:

Lemma 1.2.2. Let $a_1, a_2, \ldots, a_n$ be n positive reals. Then,

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq \frac{n^2}{a_1 + a_2 + \cdots + a_n}.$$

Proof of Lemma 1.2.2. There are several ways to prove this:

- Apply Jensen's inequality to the convex function $\mathbb{R}^+ \to \mathbb{R}^+,\ x \mapsto \frac{1}{x}$.
- Apply the Cauchy-Schwarz inequality to get

$$(a_1 + a_2 + \cdots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right)$$
$$\geq \left(\underbrace{\sqrt{a_1 \frac{1}{a_1}} + \sqrt{a_2 \frac{1}{a_2}} + \cdots + \sqrt{a_n \frac{1}{a_n}}}_{=n} \right)^2 = n^2.$$

- Apply the AM-HM inequality.
- Apply the AM-GM inequality twice, then multiply.
- There is a direct proof, too: First, recall the famous inequality

$$\frac{u}{v} + \frac{v}{u} \geq 2, \tag{5}$$

which holds for any two positive reals u and v. (This follows by observing that $\frac{u}{v} + \frac{v}{u} - 2 = \frac{(u-v)^2}{uv} \geq 0$.) Now,

$$\begin{aligned}
&(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \\
&= \left(\sum_{i=1}^n a_i\right)\left(\sum_{j=1}^n \frac{1}{a_j}\right) = \sum_{i=1}^n \sum_{j=1}^n a_i \frac{1}{a_j} = \sum_{i=1}^n \sum_{j=1}^n \frac{a_i}{a_j} \\
&= \frac{1}{2}\left(\sum_{i=1}^n \sum_{j=1}^n \frac{a_i}{a_j} + \sum_{i=1}^n \sum_{j=1}^n \frac{a_i}{a_j}\right) \qquad \left(\text{since } x = \frac{1}{2}(x+x) \text{ for any } x \in \mathbb{R}\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^n \sum_{j=1}^n \frac{a_i}{a_j} + \sum_{j=1}^n \sum_{i=1}^n \frac{a_j}{a_i}\right) \qquad \left(\begin{array}{c}\text{here, we renamed } i \text{ and } j \text{ as } j \text{ and } i \\ \text{in the second double sum}\end{array}\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^n \sum_{j=1}^n \frac{a_i}{a_j} + \sum_{i=1}^n \sum_{j=1}^n \frac{a_j}{a_i}\right) \qquad \left(\begin{array}{c}\text{here, we swapped the two} \\ \text{summation signs in the} \\ \text{second double sum}\end{array}\right) \\
&= \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \underbrace{\left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right)}_{\substack{\geq 2 \\ \text{(by (5))}}} \geq \frac{1}{2}\underbrace{\sum_{i=1}^n \sum_{j=1}^n 2}_{=n^2 \cdot 2} = \frac{1}{2}n^2 \cdot 2 = n^2,
\end{aligned}$$

from which the claim of Lemma 1.2.2 follows.

□

Proof of Corollary 1.2.1. Write the multigraph G as $G = (V, E, \varphi)$. Thus, $|V| = n$ and $|E| = m$. We WLOG assume that $V = \{1, 2, \ldots, n\}$ (since $|V| = n$). Hence,

$$\begin{aligned}
\sum_{v=1}^n \deg v = \sum_{v \in V} \deg v = 2 \cdot \underbrace{|E|}_{=m} \qquad &\text{(by Proposition 1.1.3 in Lecture 2)} \\
= 2m.
\end{aligned}$$

However, Theorem 1.1.3 yields that G has an independent set of size

$$\begin{aligned}
&\geq \sum_{v\in V} \frac{1}{1+\deg v} = \sum_{v=1}^{n} \frac{1}{1+\deg v} \qquad (\text{since } V = \{1,2,\ldots,n\}) \\
&\geq \frac{n^2}{\sum\limits_{v=1}^{n} (1+\deg v)} \qquad \left(\begin{array}{c}\text{by Lemma 1.2.2, applied to the } n \text{ positive} \\ \text{reals } a_v = 1+\deg v \text{ for all } v \in \{1,2,\ldots,n\}\end{array}\right) \\
&= \frac{n^2}{n+2m} \qquad \left(\text{since } \sum_{v=1}^{n} (1+\deg v) = n + \underbrace{\sum_{v\in V} \deg v}_{=2m} = n+2m\right).
\end{aligned}$$

This proves Corollary 1.2.1. □

1.3. A proof of Turan's theorem

Recall Turan's theorem, which we stated but did not prove in Lecture 2:

> **Theorem 1.3.1** (Turan's theorem)**.** Let r be a positive integer. Let G be a simple graph with n vertices and e edges. Assume that
>
> $$e > \frac{r-1}{r} \cdot \frac{n^2}{2}.$$
>
> Then, there exist $r+1$ distinct vertices of G that are mutually adjacent (i.e., any two distinct vertices among these $r+1$ vertices are adjacent).

We can now easily derive it from Corollary 1.2.1:

Proof of Theorem 1.3.1. Write the simple graph G as $G = (V, E)$. Thus, $|V| = n$ and $|E| = e$ and $E \subseteq \mathcal{P}_2(V)$.

Let $E' := \mathcal{P}_2(V) \setminus E$. Thus, the set E' consists of all "non-edges" of G – that is, of all 2-element subsets of V that are not edges of G. Clearly,

$$|E'| = |\mathcal{P}_2(V) \setminus E| = \underbrace{|\mathcal{P}_2(V)|}_{=\binom{n}{2}} - \underbrace{|E|}_{=e} = \binom{n}{2} - e.$$

Now, let G' be the simple graph (V, E'). This simple graph G' is called the

complementary graph of G; it has n vertices and $|E'| = \binom{n}{2} - e$ edges.[7] Hence, Corollary 1.2.1 (applied to G' and $\binom{n}{2} - e$ instead of G and m) yields that G' has an independent set of size

$$\geq \frac{n^2}{n + 2 \cdot \left(\binom{n}{2} - e\right)}.$$

Let S be this independent set. Its size is

$$|S| \geq \frac{n^2}{n + 2 \cdot \left(\binom{n}{2} - e\right)} = \frac{n^2}{n + n\,(n-1) - 2e} = \frac{n^2}{n^2 - 2e} > r$$

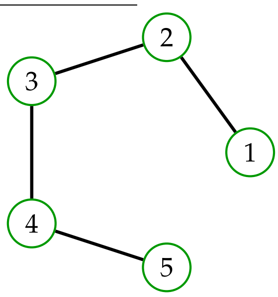
(this inequality follows by high-school algebra from $e > \frac{r-1}{r} \cdot \frac{n^2}{2}$). Hence, $|S| \geq r + 1$ (since $|S|$ and r are integers). However, S is an independent set of G'. Thus, any two distinct vertices in S are non-adjacent in G' and therefore adjacent in G (by the definition of G'). Since $|S| \geq r + 1$, we have thus found $r + 1$ (or more) distinct vertices of G that are mutually adjacent in G. This proves Theorem 1.3.1. □

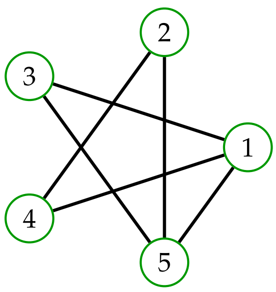
Several other beautiful proofs of Theorem 1.3.1 can be found in [AigZie18, Chapter 41].

References

[AigZie18] Martin Aigner, Günter M. Ziegler, *Proofs from THE BOOK*, 6th edition, Springer 2018.

[Aspnes23] James Aspnes, *Notes on Randomized Algorithms (CPSC 469/569: Spring 2023)*, 1 May 2023.
`https://www.cs.yale.edu/homes/aspnes/classes/469/notes.pdf`

[7]For example, if $G =$ 

, then $G' =$ 

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