Math 530 Spring 2022, Lecture 22: More on colorings

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Colorings

1.1. 2-colorings

Last time, we stated but didn't prove the following theorem:

Theorem 1.1.1 (2-coloring equivalence theorem)**.** Let $G = (V, E, \varphi)$ be a multigraph. Then, the following three statements are equivalent:

- **Statement B1:** The graph G has a proper 2-coloring.
- **Statement B2:** The graph G has no cycles of odd length.
- **Statement B3:** The graph G has no circuits of odd length.

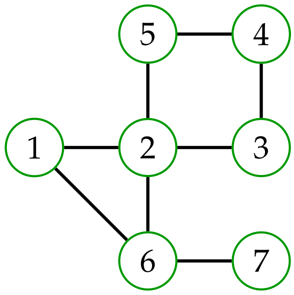
Let us see how this theorem is proved. We will need a fact that is somewhat similar to Proposition 1.1.9 in Lecture 8:

Proposition 1.1.2. Let G be a multigraph. Let u and v be two vertices of G. Let $\mathbf{w}$ be an odd-length walk from u to v. Then, $\mathbf{w}$ contains either an odd-length **path** from u to v or an odd-length **cycle** (or both).

Here, we are using the following rather intuitive terminology:

- A walk is said to be **odd-length** if its length is odd.
- A walk $\mathbf{w}$ is said to **contain** a walk $\mathbf{v}$ if each edge of $\mathbf{v}$ is an edge of $\mathbf{w}$. (This does not necessarily mean that $\mathbf{v}$ appears in $\mathbf{w}$ as a contiguous block.)
- We remind the reader once again that a "circuit" just means a closed walk to us; we impose no further requirements.

Example 1.1.3. Consider the following simple graph (which we treat as a multigraph):



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(a) The odd-length walk $(1, *, 2, *, 3, *, 4, *, 5, *, 2, *, 6, *, 7)$ (we are using asterisks for the edges, since they can be trivially recovered from the vertices) contains the odd-length path $(1, *, 2, *, 6, *, 7)$ from 1 to 7.

(b) The odd-length walk $(3, *, 2, *, 1, *, 6, *, 2, *, 3)$ contains the odd-length cycle $(2, *, 1, *, 6, *, 2)$.

Proof of Proposition 1.1.2. We apply strong induction on the length of $\mathbf{w}$.

Thus, we fix a $k \in \mathbb{N}$, and we assume (as the induction hypothesis) that Proposition 1.1.2 is already proved for all odd-length walks of length $< k$. Now, we must prove it for an odd-length walk $\mathbf{w}$ of length k.

Write this walk $\mathbf{w}$ as $\mathbf{w} = (w_0, *, w_1, *, w_2, \ldots, *, w_k)$. Hence, k is the length of $\mathbf{w}$, and thus is odd.

We must prove that $\mathbf{w}$ contains either an odd-length path from u to v or an odd-length cycle.

If $\mathbf{w}$ itself is a path, then we are done. So WLOG assume that $\mathbf{w}$ is not a path. Thus, two of the vertices $w_0, w_1, \ldots, w_k$ of $\mathbf{w}$ are equal. In other words, there exists a pair (i, j) of integers i and j with $0 \leq i < j \leq k$ and $w_i = w_j$. Among all such pairs, we pick one with **minimum** difference $j - i$. Then, the vertices $w_i, w_{i+1}, \ldots, w_{j-1}$ are distinct (since $j - i$ is minimum).

Let $\mathbf{c}$ be the part of $\mathbf{w}$ between w_i and w_j; thus,[1]

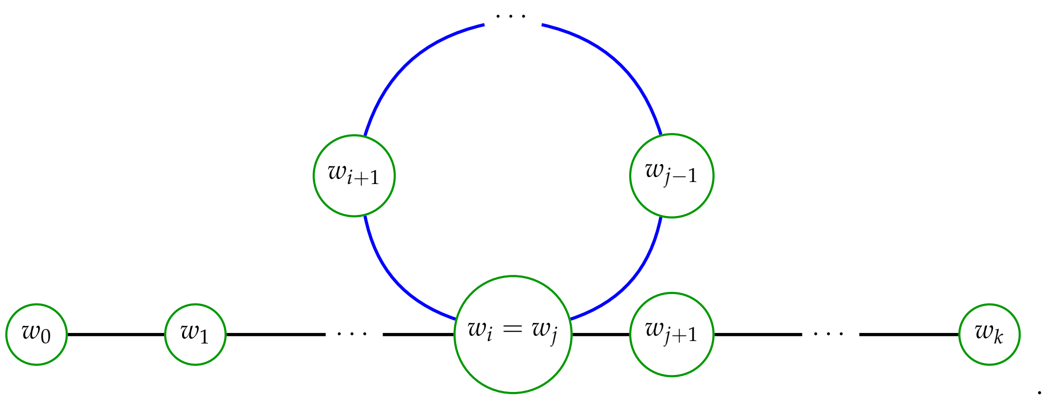
$$\mathbf{c} = (w_i, *, w_{i+1}, *, \ldots, *, w_j).$$

This $\mathbf{c}$ is clearly a closed walk (since $w_i = w_j$). If $j - i$ is odd, then this closed walk $\mathbf{c}$ is a cycle (indeed, its vertices $w_i, w_{i+1}, \ldots, w_{j-1}$ are distinct, and therefore its edges are distinct as well[2]), and thus we have found an odd-length cycle contained in $\mathbf{w}$ (namely, $\mathbf{c}$ is such a cycle, since its length is $j - i$, which is odd). This means that we are done if $j - i$ is odd.

Thus, we WLOG assume that $j - i$ is even. Hence, cutting out the closed walk $\mathbf{c}$ from the original walk $\mathbf{w}$, we obtain a walk

$$\mathbf{w}' := (w_0, *, w_1, *, \ldots, *, w_i = w_j, *, w_{j+1}, *, w_{j+2}, \ldots, w_k)$$

[1] Here is an illustration (which, however, is a bit simplistic: the walk $\mathbf{w}$ can intersect itself arbitrarily many times, not just once as shown here):



The blue edges here form the walk $\mathbf{c}$.

[2] For the very skeptical, here is a *proof* of this:

Assume (for the sake of contradiction) that the walk $\mathbf{c}$ has two equal edges. Let the first of them be an edge between w_p and w_{p+1}, and let the second be an edge between w_q and w_{q+1}, for some distinct elements p and q of $\{i, i+1, \ldots, j-1\}$. Since equal edges have equal endpoints, we thus have $\{w_p, w_{p+1}\} = \{w_q, w_{q+1}\}$, so that $w_p \in \{w_p, w_{p+1}\} = \{w_q, w_{q+1}\}$. In other words, w_p equals either w_q or w_{q+1}. Since $w_p \neq w_q$ (because $w_i, w_{i+1}, \ldots, w_{j-1}$ are distinct), this entails that $w_p = w_{q+1}$. Similarly, $w_q = w_{p+1}$.

However, p and q are distinct. Thus, at least one of p and q is distinct from $j - 1$. We WLOG assume that $q \neq j - 1$ (otherwise, we can swap p with q). Hence, $q + 1 \neq j$, so that $q + 1 \in \{i, i+1, \ldots, j-1\}$. Thus, from $w_p = w_{q+1}$, we conclude that $p = q + 1$ (since $w_i, w_{i+1}, \ldots, w_{j-1}$ are distinct). Thus, $p = q + 1 > q$, so that $p + 1 > p > q$ and therefore $p + 1 \neq q$. However, $w_q = w_{p+1}$. If $p + 1$ was an element of $\{i, i+1, \ldots, j-1\}$, then this would entail $q = p + 1$ (since $w_i, w_{i+1}, \ldots, w_{j-1}$ are distinct), which would contradict $p + 1 \neq q$. Thus, $p + 1$ cannot be an element of $\{i, i+1, \ldots, j-1\}$. Hence, $p + 1 = j$ (since $p + 1$ clearly belongs to $\{i, i+1, \ldots, j\}$). Thus, $w_{p+1} = w_j = w_i$, so that $w_i = w_{p+1} = w_q$. This entails $i = q$ (since $w_i, w_{i+1}, \ldots, w_{j-1}$ are distinct). Hence, $i = q = p - 1$ (since $p = q + 1$). Therefore, $\underbrace{j}_{=p+1} - \underbrace{i}_{=p-1} = (p+1) - (p-1) = 2$. This contradicts the fact that $j - i$ is odd.

This contradiction shows that our assumption (that the walk $\mathbf{c}$ has two equal edges) was false. Hence, the edges of $\mathbf{c}$ are distinct.

from u to v. This new walk $\mathbf{w}'$ has length $k-(j-i)$, which is odd (since k is odd but $j-i$ is even) and smaller than k (since $i<j$). Hence, we can apply the induction hypothesis to this walk $\mathbf{w}'$. As a consequence, we conclude that this walk $\mathbf{w}'$ contains either an odd-length path from u to v or an odd-length cycle. Therefore, the walk $\mathbf{w}$ also contains either an odd-length path from u to v or an odd-length cycle (since anything contained in $\mathbf{w}'$ is automatically contained in $\mathbf{w}$). But this is precisely what we set out to prove. This completes the induction step, and so we have proved Proposition 1.1.2. □

Now, let us prove the 2-coloring equivalence theorem:

Proof of Theorem 1.1.1. We shall prove the implications B1 $\Longrightarrow$ B2 $\Longrightarrow$ B3 $\Longrightarrow$ B1.

Proof of the implication B1 $\Longrightarrow$ B2: Assume that Statement B1 holds. We must prove that Statement B2 holds.

We have assumed that B1 holds. In other words, the graph G has a proper 2-coloring. Let f be this 2-coloring. Thus, f is a map from V to $\{1,2\}$ such that any two adjacent vertices x and y of G satisfy $f(x) \neq f(y)$.

Assume (for contradiction) that G has a cycle of odd length. Let

$$(v_0, *, v_1, *, v_2, *, \ldots, *, v_k)$$

be this cycle. Thus, k is odd, and we have $v_k = v_0$, so that $f(v_k) = f(v_0)$. Moreover, for each $i \in \{1,2,\ldots,k\}$, the vertex v_i is adjacent to v_{i-1} (since $(v_0, *, v_1, *, v_2, *, \ldots, *, v_k)$ is a cycle) and therefore satisfies

$$f(v_i) \neq f(v_{i-1}) \qquad (1)$$

(since f is a proper 2-coloring).

We WLOG assume that $f(v_0) = 1$ (otherwise, we "rename" the colors 1 and 2 so that the color $f(v_0)$ becomes 1). Then, (1) (applied to $i=1$) yields $f(v_1) \neq f(v_0) = 1$, so that $f(v_1) = 2$ (since $f(v_1)$ must be either 1 or 2). Hence, (1) (applied to $i=2$) yields $f(v_2) \neq f(v_1) = 2$, so that $f(v_2) = 1$ (since $f(v_2)$ must be either 1 or 2). For similar reasons, we can successively obtain $f(v_3) = 2$ and $f(v_4) = 1$ and $f(v_5) = 2$ and so on. The general formula we obtain (strictly speaking, it needs to be proved by induction on i) says that

$$f(v_i) = \begin{cases} 1, & \text{if } i \text{ is even;} \\ 2, & \text{if } i \text{ is odd} \end{cases} \qquad \text{for each } i \in \{0,1,\ldots,k\}.$$

Applying this to $i=k$, we conclude that $f(v_k) = 2$ (since k is odd). However, this contradicts $f(v_k) = f(v_0) = 1 \neq 2$. This contradiction shows that our assumption was false. Hence, G has no cycle of odd length. In other words, Statement B2 holds. This proves the implication B1 $\Longrightarrow$ B2.

Proof of the implication B2 $\Longrightarrow$ B3: Assume that Statement B2 holds. We must prove that Statement B3 holds. In other words, we must show that G has no odd-length circuits.

Assume the contrary. Thus, G has an odd-length circuit $\mathbf{w}$. Let u be the starting and ending point of $\mathbf{w}$. Thus, Proposition 1.1.2 (applied to $v = u$) shows that this odd-length circuit $\mathbf{w}$ contains either an odd-length **path** from u to u or an odd-length cycle. Since G has no odd-length cycle (because we assumed that Statement B2 holds), we thus concludes that $\mathbf{w}$ contains an odd-length **path** from u to u. However, an odd-length path from u to u is impossible (since the only path from u to u has length 0). Thus, we obtain a contradiction, which shows that G has no odd-length circuits. This proves the implication B2 $\Longrightarrow$ B3.

Proof of the implication B3 $\Longrightarrow$ B1: Assume that Statement B3 holds. We must prove that Statement B1 holds.

We have assumed that Statement B3 holds. In other words, G has no odd-length circuits. We must find a proper 2-coloring of G.

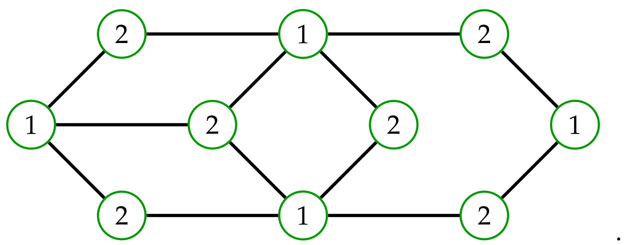
We WLOG assume that G is connected (otherwise, let $C_1, C_2, \ldots, C_k$ be the components of G, and apply the implication B3 $\Longrightarrow$ B1 to each of the smaller graphs $G[C_1]$, $G[C_2]$, $\ldots$, $G[C_k]$, and then combine the resulting proper 2-colorings of these smaller graphs into a single proper 2-coloring of G). Fix any vertex r of G. Define a map $f : V \to \{1, 2\}$ by setting

$$f(v) = \begin{cases} 1, & \text{if } d(v,r) \text{ is even;} \\ 2, & \text{if } d(v,r) \text{ is odd} \end{cases} \qquad \text{for each } v \in V$$

(where $d(v,r)$ denotes the distance from v to r, that is, the smallest length of a path from v to r).

I claim that f is a proper 2-coloring.[3] Indeed, assume the contrary. Thus, some two adjacent vertices u and v have the same color $f(u) = f(v)$. Consider these u and v. Since $f(u) = f(v)$, we are in one of the following two cases:

[3]Here is an illustrative example:



(Of course, the numbers on the nodes here are not the vertices, but rather the colors of these vertices.)

Note that all values of f can be easily found by the following recursive algorithm: Start by assigning the color 1 to r. Then, assign the color 2 to all neighbors of r. Then, assign the color 1 to all neighbors of these neighbors (unless they have already been colored). Then, assign the color 2 to all neighbors of these neighbors of these neighbors, and so on.

Case 1: We have $f(u) = f(v) = 1$.

Case 2: We have $f(u) = f(v) = 2$.

Let us consider Case 2. In this case, we have $f(u) = f(v) = 2$. This means that $d(u,r)$ and $d(v,r)$ are both odd (by the definition of f). Hence, there is an odd-length path $\mathbf{p}$ from u to r and an odd-length path $\mathbf{q}$ from v to r. Consider these $\mathbf{p}$ and $\mathbf{q}$. Also, there is an edge e that joins u and v (since u and v are adjacent). Consider this edge e. By combining the paths $\mathbf{p}$ and $\mathbf{q}$ and inserting the edge e into the result, we obtain a circuit from r to r (which starts by following the path $\mathbf{p}$ backwards to u, then takes the edge e to v, then follows the path $\mathbf{q}$ back to r). This circuit has odd length (since $\mathbf{p}$ and $\mathbf{q}$ have odd lengths, and since the edge e adds 1 to the length). Thus, we have found an odd-length circuit of G. However, we assumed that G has no odd-length circuits. Contradiction!

Thus, we have found a contradiction in Case 2. Similarly, we can find a contradiction in Case 1. Thus, we always get a contradiction. This shows that f is indeed a proper 2-coloring. Thus, Statement B1 holds. This proves the implication B3 $\Longrightarrow$ B1. [4]

For aesthetical reasons, let me give a *second proof of the implication B3 $\Longrightarrow$ B1*, which avoids the awkward "break G up into components" step:

Assume again that Statement B3 holds. We must prove that Statement B1 holds.

We assumed that Statement B3 holds. In other words, G has no odd-length cycles.

Two vertices u and v of G will be called **oddly connected** if G has an odd-length path from u to v. By Proposition 1.1.2, this condition is equivalent to "G has an odd-length walk from u to v", since G has no odd-length cycles. Moreover, a vertex u cannot be oddly connected to itself (since the only path from u to u is the trivial length-0 path (u), which is not odd-length).

A subset A of V will be called **odd-path-less** if no two vertices in A are oddly connected. (Note that "two vertices" doesn't mean "two distinct vertices".)

Pick a maximum-size odd-path-less subset A of V (such an A exists, since $\varnothing$ is clearly odd-path-less). Now, let $f : V \to \{1,2\}$ be the 2-coloring of G that assigns the color 1 to all vertices in A and assigns the color 2 to all vertices not in A.

We shall show that this 2-coloring f is proper.

To prove this, we must show that no two adjacent vertices have color 1 and that no two adjacent vertices have color 2. The first of these two claims is obvious[5]. It thus remains to prove the second claim – i.e., to prove that no two adjacent vertices have color 2.

Assume the contrary. Thus, there exist two adjacent vertices u and v that both have

[4] Note that this proof provides a reasonably efficient algorithm for constructing a proper 2-coloring of G, as long as you know how to compute distances in a graph (we have done this, e.g., in homework set #4 exercise 5) and how to compute the components of a graph (this is not hard).

[5] *Proof.* An edge always makes a walk of length 1, which is odd. Thus, two adjacent vertices are automatically oddly connected. Hence, two adjacent vertices cannot both be contained in the odd-path-less subset A. In other words, two adjacent vertices cannot both have color 1.

color 2. Consider these u and v. These vertices u and v have color 2; in other words, neither of them belongs to A.

The vertex u is not oddly connected to itself (as we already saw). Hence, the vertex u is oddly connected to at least one vertex $a \in A$ (because otherwise, we could insert u into the odd-path-less set A and obtain a larger odd-path-less subset $A \cup \{u\}$ of V; but this would contradict the fact that A is a **maximum-size** odd-path-less subset of V). For similar reasons, the vertex v is oddly connected to at least one vertex $b \in A$. Consider these vertices a and b. Since u is oddly connected to a, there exists an odd-length walk $\mathbf{p}$ from u to a. Reversing this walk $\mathbf{p}$ yields an odd-length walk $\mathbf{p}'$ from a to u. Since v is oddly connected to b, there exists an odd-length walk $\mathbf{q}$ from v to b. Finally, there is an edge e with endpoints u and v (since u and v are adjacent). Combine the two walks $\mathbf{p}'$ and $\mathbf{q}$ and insert this edge e between them; this yields a walk from a to b (via u and v) that has odd length (since $\mathbf{p}'$ and $\mathbf{q}$ have odd length each, and inserting e adds 1 to the length). Thus, G has an odd-length walk from a to b. In other words, the vertices a and b are oddly connected. This contradicts the fact that the set A is odd-path-less (since a and b belong to A).

This contradiction shows that our assumption was false. Thus, we have shown that no two adjacent vertices have color 2. This completes our proof that f is a proper 2-coloring. Thus, Statement B1 holds. This proves the implication B3 $\Longrightarrow$ B1 once again.

Having proved all three implications B1 $\Longrightarrow$ B2 and B2 $\Longrightarrow$ B3 and B3 $\Longrightarrow$ B1, we now conclude that the three statements B1, B2 and B3 are equivalent. This proves Theorem 1.1.1. □

Remark 1.1.4. A graph G that satisfies the three equivalent statements B1, B2, B3 of Theorem 1.1.1 is sometimes called a "bipartite graph". This is slightly imprecise, since the proper definition of a "bipartite graph" is (equivalent to) "a graph **equipped with** a proper 2-coloring". Thus, if we equip one and the same graph G with different proper 2-colorings, then we obtain different bipartite graphs. We shall take a closer look at bipartite graphs in Lecture 24 and Lecture 25.

A further simple property of proper 2-colorings is the following:[6]

Proposition 1.1.5. Let G be a multigraph that has a proper 2-coloring. Then, G has exactly $2^{\operatorname{conn} G}$ many proper 2-colorings.

Proof sketch. For each component C of G, let us fix an arbitrary vertex $r_C \in C$. When constructing a proper 2-coloring f of G, we can freely choose the colors $f(r_C)$ of these vertices r_C; the colors of all other vertices are then uniquely determined (see the first proof of the implication B3 $\Longrightarrow$ B1 in our above proof of Theorem 1.1.1 for the details). Thus, we have $2^{\operatorname{conn} G}$ many options (since G has $\operatorname{conn} G$ many components). The proposition follows. □

[6]Recall that $\operatorname{conn} G$ denotes the number of components of a graph G.

1.2. The Brooks theorems

As we said, the existence of a proper k-coloring for a given graph G is a hard computational problem unless $k \leq 2$. The same holds for theoretical criteria: For $k > 2$, I am not aware of any good criteria that are simultaneously necessary and sufficient for the existence of a proper k-coloring. However, some sufficient criteria are known. Here is one:[7]

Theorem 1.2.1 (Little Brooks theorem)**.** Let $G = (V, E, \varphi)$ be a loopless multigraph with at least one vertex. Let

$$\alpha := \max \{\deg v \mid v \in V\}.$$

Then, G has a proper $(\alpha + 1)$-coloring.

Proof sketch. Let $v_1, v_2, \ldots, v_n$ be the vertices of V, listed in some order (with no repetitions). We construct a proper $(\alpha + 1)$-coloring $f : V \to \{1, 2, \ldots, \alpha + 1\}$ of G recursively as follows:

- First, we choose $f(v_1)$ arbitrarily.
- Then, we choose $f(v_2)$ to be distinct from the colors of all already-colored neighbors of v_2.
- Then, we choose $f(v_3)$ to be distinct from the colors of all already-colored neighbors of v_3.
- Then, we choose $f(v_4)$ to be distinct from the colors of all already-colored neighbors of v_4.
- And so on, until all values $f(v_1), f(v_2), \ldots, f(v_n)$ have been chosen.

Why do we never run out of colors in this process? Well: When choosing $f(v_i)$, we must choose a color distinct from the colors of all already-colored neighbors of v_i. Since v_i has at most α neighbors (because $\deg(v_i) \leq \alpha$), this means that we have at most α colors to avoid. Since there are $\alpha + 1$ colors in total, this leaves us at least 1 color that we can choose; therefore, we don't run out of colors.

The resulting $(\alpha + 1)$-coloring $f : V \to \{1, 2, \ldots, \alpha + 1\}$ is called a **greedy coloring**. This $(\alpha + 1)$-coloring f is indeed proper, because if an edge has endpoints v_i and v_j with $i > j$, then the construction of $f(v_i)$ ensures that $f(v_i)$ is distinct from $f(v_j)$. (Note how we are using the fact that G is loopless here! If G had a loop, then the endpoints of this loop could not be written as v_i and v_j with $i > j$.) $\square$

[7] Recall that a multigraph is called **loopless** if it has no loops.

In general, the $\alpha + 1$ in Theorem 1.2.1 cannot be improved. Here are two examples:

- If $n \geq 2$, then the cycle graph C_n [8] has maximum degree $\alpha = \max\{\deg v \mid v \in V\} = 2$. Thus, Theorem 1.2.1 shows that C_n has a proper 3-coloring. When n is even, C_n has a proper 2-coloring as well, but this is not the case when n is odd (by Theorem 1.1.1).
- If $n \geq 1$, then the complete graph K_n has maximum degree $\alpha = \max\{\deg v \mid v \in V\} = n - 1$. Thus, Theorem 1.2.1 shows that K_n has a proper n-coloring. By the pigeonhole principle, it is clear that K_n has no proper $(n-1)$-coloring.

Interestingly, these two examples are in fact the only cases when a connected loopless multigraph with maximum degree α can fail to have a proper α-coloring. In all other cases, we can improve the $\alpha + 1$ to α:

Theorem 1.2.2 (Brooks theorem)**.** Let $G = (V, E, \varphi)$ be a connected loopless multigraph. Let

$$\alpha := \max\{\deg v \mid v \in V\}.$$

Assume that G is neither a complete graph nor an odd-length cycle. Then, G has a proper α-coloring.

Proof. Despite the seemingly little difference, this is significantly harder to prove than Theorem 1.2.1. Various proofs can be found in [CraRab15] and in most serious textbooks on graph theory. □

1.3. The chromatic polynomial

Here is another surprise: The number of proper k-colorings of a given multigraph G turns out to be a polynomial function in k (with integer coefficients). More precisely:

Theorem 1.3.1 (Whitney's chromatic polynomial theorem)**.** Let $G = (V, E, \varphi)$ be a multigraph. Let χ_G be the polynomial in the single indeterminate x with coefficients in $\mathbb{Z}$ defined as follows:

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V, F, \varphi|_F)} = \sum_{\substack{H \text{ is a spanning} \\ \text{subgraph of } G}} (-1)^{|\mathrm{E}(H)|} x^{\operatorname{conn} H}.$$

(The symbol "$\sum_{F \subseteq E}$" means "sum over all subsets F of E".)

Then, for any $k \in \mathbb{N}$, we have

$$(\text{\# of proper } k\text{-colorings of } G) = \chi_G(k).$$

[8] See Definition 1.3.9 in Lecture 7 for the proper definition of C_n when $n = 2$.

The proper place for this theorem is probably a course on enumerative combinatorics, but let us give here a proof for the sake of completeness (optional material). The following proof is essentially due to Hassler Whitney in 1930 ([Whitne32, §6]), and I am mostly copypasting it from my own writeup [17s-mt2s, §0.5] (with some changes stemming from the fact that we are here working with multigraphs rather than simple graphs).

We are going to use the **Iverson bracket notation**:

Definition 1.3.2. If $\mathcal{A}$ is any logical statement, then $[\mathcal{A}]$ shall denote the truth value of $\mathcal{A}$; this is the number $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$

For instance, $[2+2=4] = 1$ and $[2+2=5] = 0$.

We next recall a combinatorial identity ([Grinbe17, Lemma 3.3.5]):

Lemma 1.3.3. Let P be a finite set. Then,

$$\sum_{A \subseteq P} (-1)^{|A|} = [P = \varnothing].$$

(The symbol " $\sum\limits_{A \subseteq P}$ " means "sum over all subsets A of P".)

Next, we introduce a specific notation related to colorings:

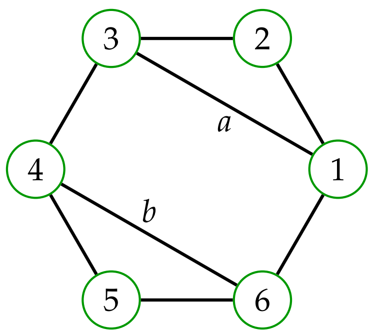
Definition 1.3.4. Let $G = (V, E, \varphi)$ be a multigraph. Let $k \in \mathbb{N}$. Let $f : V \to \{1, 2, \ldots, k\}$ be a k-coloring. We then define a subset E_f of E by

$$E_f := \{e \in E \mid \text{the two endpoints of } e \text{ have the same color in } f\}.$$

(Recall that the "color in f" of a vertex v means the value $f(v)$. If an edge $e \in E$ is a loop, then e always belongs to E_f, since we think of the two endpoints of e as being equal.)

The elements of E_f are called the ***f*-monochromatic** edges of G. ("Monochromatic" means "one-colored", so no surprises here.)

Example 1.3.5. Let $G = (V, E, \varphi)$ be the following multigraph:



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Let $f : V \to \{1, 2\}$ be the 2-coloring of G that sends each odd vertex to 1 and each even vertex to 2. (Here, an "odd vertex" means a vertex that is odd as an integer. Thus, the odd vertices are $1, 3, 5$. "Even vertices" are understood similarly.) Then, $E_f = \{a, b\}$.

Notice the following simple fact:

Proposition 1.3.6. Let $G = (V, E, \varphi)$ be a multigraph. Let $k \in \mathbb{N}$. Let $f : V \to \{1, 2, \ldots, k\}$ be a k-coloring. Then, the k-coloring f is proper if and only if $E_f = \varnothing$.

Proof of Proposition 1.3.6. We have the following chain of equivalences:

$$\begin{aligned}
& (\text{the } k\text{-coloring } f \text{ is proper}) \\
\Longleftrightarrow\ & (\text{no two adjacent vertices have the same color}) \\
& \qquad (\text{by the definition of "proper"}) \\
\Longleftrightarrow\ & (\text{there is no edge } e \in E \text{ such that the two endpoints of } e \text{ have the same color}) \\
& \qquad \left(\begin{array}{c}\text{since adjacent vertices are vertices that} \\ \text{are the two endpoints of an edge}\end{array}\right) \\
\Longleftrightarrow\ & \left(\text{there exists no element of } E_f\right) \\
& \qquad \left(\begin{array}{c}\text{since the elements of } E_f \text{ are precisely the edges } e \in E \\ \text{such that the two endpoints of } e \text{ have the same color} \\ (\text{by the definition of } E_f)\end{array}\right) \\
\Longleftrightarrow\ & \left(E_f = \varnothing\right).
\end{aligned}$$

This proves Proposition 1.3.6. □

Lemma 1.3.7. Let $G = (V, E, \varphi)$ be a multigraph. Let B be a subset of E. Let $k \in \mathbb{N}$. Then, the number of all k-colorings $f : V \to \{1, 2, \ldots, k\}$ satisfying $B \subseteq E_f$ is $k^{\operatorname{conn}(V, B, \varphi|_B)}$.

Proof of Lemma 1.3.7. If C is a nonempty subset of V, and if $f : V \to \{1, 2, \ldots, k\}$ is any k-coloring of G, then we shall say that f is **constant on** C if the restriction $f\mid_C$ is a constant map (i.e., if the colors $f(c)$ for all $c \in C$ are equal). We shall show the following claim:

> *Claim 1:* Let $f : V \to \{1, 2, \ldots, k\}$ be any k-coloring of G. Then, we have $B \subseteq E_f$ if and only if f is constant on each component of the multigraph $(V, B, \varphi\mid_B)$.

[*Proof of Claim 1:* This is an "if and only if" statement; we shall prove its "$\Longrightarrow$" and "$\Longleftarrow$" directions separately:

$\Longrightarrow$: Assume that $B \subseteq E_f$. We must prove that f is constant on each component of the multigraph $(V, B, \varphi \mid_B)$.

Let C be a component of $(V, B, \varphi \mid_B)$. We must prove that f is constant on C. In other words, we must prove that $f(c) = f(d)$ for any $c, d \in C$.

So let us fix $c, d \in C$. Then, the vertices c and d belong to the same component of the graph $(V, B, \varphi \mid_B)$ (namely, to C). Hence, these vertices c and d are path-connected in this graph. In other words, the graph $(V, B, \varphi \mid_B)$ has a path from c to d. Let

$$\mathbf{p} = (v_0, e_1, v_1, e_2, v_2, \ldots, e_s, v_s)$$

be this path. Hence, $v_0 = c$ and $v_s = d$ and $e_1, e_2, \ldots, e_s \in B$.

Let $i \in \{1, 2, \ldots, s\}$. Then, the endpoints of the edge e_i are v_{i-1} and v_i (since e_i is surrounded by v_{i-1} and v_i on the path $\mathbf{p}$). However, from $e_1, e_2, \ldots, e_s \in B$, we obtain $e_i \in B \subseteq E_f$. Hence, the two endpoints of e_i have the same color in f (by the definition of E_f). In other words, $f(v_{i-1}) = f(v_i)$ (since the endpoints of the edge e_i are v_{i-1} and v_i).

Forget that we fixed i. We thus have proved the equality $f(v_{i-1}) = f(v_i)$ for each $i \in \{1, 2, \ldots, s\}$. Combining these equalities, we obtain

$$f(v_0) = f(v_1) = f(v_2) = \cdots = f(v_s).$$

Hence, $f(v_0) = f(v_s)$. In other words, $f(c) = f(d)$ (since $v_0 = c$ and $v_s = d$).

Forget that we fixed c and d. We thus have shown that $f(c) = f(d)$ for any $c, d \in C$. In other words, f is constant on C. Since C was allowed to be an arbitrary component of $(V, B, \varphi \mid_B)$, we thus conclude that f is constant on each component of the multigraph $(V, B, \varphi \mid_B)$. This proves the "$\Longrightarrow$" direction of Claim 1.

$\Longleftarrow$: Assume that f is constant on each component of the multigraph $(V, B, \varphi \mid_B)$. We must prove that $B \subseteq E_f$.

Indeed, let $e \in B$. Let u and v be the two endpoints of e. Then, (u, e, v) is a walk from u to v in the multigraph $(V, B, \varphi \mid_B)$ (since $e \in B$). Hence, u is path-connected to v in this multigraph. In other words, u and v belong to the same component of the multigraph $(V, B, \varphi \mid_B)$. Therefore, $f(u) = f(v)$ (since f is constant on each component of the multigraph $(V, B, \varphi \mid_B)$). This means that the two endpoints of e have the same color in f (since u and v are the endpoints of e). Combining this with the fact that $e \in E$ (because $e \in B \subseteq E$), we conclude that $e \in E_f$ (by the definition of E_f).

Forget that we fixed e. We thus have shown that $e \in E_f$ for each $e \in B$. In other words, $B \subseteq E_f$. This proves the "$\Longleftarrow$" direction of Claim 1. The proof of Claim 1 is now complete.]

Now, Claim 1 shows that the k-colorings $f : V \to \{1, 2, \ldots, k\}$ satisfying $B \subseteq E_f$ are precisely the k-colorings $f : V \to \{1, 2, \ldots, k\}$ that are constant on each component of the graph $(V, B, \varphi \mid_B)$. Hence, all such k-colorings f can be obtained by the following procedure:

- **For each** component C of the graph $(V, B, \varphi\mid_B)$, pick a color c_C (that is, an element c_C of $\{1, 2, \ldots, k\}$) and then assign this color c_C to each vertex in C (that is, set $f(v) = c_C$ for each $v \in C$).

This procedure involves choices (because for each component C of $(V, B, \varphi\mid_B)$, we get to pick a color): Namely, for each of the $\operatorname{conn}(V, B, \varphi\mid_B)$ many components of the graph $(V, B, \varphi\mid_B)$, we must choose a color from the set $\{1, 2, \ldots, k\}$. Thus, we have a total of $k^{\operatorname{conn}(V,B,\varphi\mid_B)}$ many options (since we are choosing among k colors for each of the $\operatorname{conn}(V, B, \varphi\mid_B)$ components). Each of these options gives rise to a different k-coloring $f : V \to \{1, 2, \ldots, k\}$. Therefore, the number of all k-colorings $f : V \to \{1, 2, \ldots, k\}$ satisfying $B \subseteq E_f$ is $k^{\operatorname{conn}(V,B,\varphi\mid_B)}$ (because all of these k-colorings can be obtained by this procedure). This proves Lemma 1.3.7. □

Corollary 1.3.8. Let (V, E, φ) be a multigraph. Let F be a subset of E. Let $k \in \mathbb{N}$. Then,

$$k^{\operatorname{conn}(V,F,\varphi\mid_F)} = \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}} 1.$$

Proof of Corollary 1.3.8. We have

$$\begin{aligned}\sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}} 1 &= \left(\text{the number of all } f : V \to \{1, 2, \ldots, k\} \text{ satisfying } F \subseteq E_f\right) \cdot 1 \\ &= \left(\text{the number of all } f : V \to \{1, 2, \ldots, k\} \text{ satisfying } F \subseteq E_f\right) \\ &= k^{\operatorname{conn}(V,F,\varphi\mid_F)}\end{aligned}$$

(because Lemma 1.3.7 (applied to $B = F$) shows that the number of all k-colorings $f : V \to \{1, 2, \ldots, k\}$ satisfying $F \subseteq E_f$ is $k^{\operatorname{conn}(V,F,\varphi\mid_F)}$). This proves Corollary 1.3.8. □

Proof of Theorem 1.3.1. First of all, the equality

$$\sum_{F\subseteq E} (-1)^{|F|}\, x^{\operatorname{conn}(V,F,\varphi\mid_F)} = \sum_{\substack{H \text{ is a spanning}\\ \text{subgraph of } G}} (-1)^{|\mathrm{E}(H)|}\, x^{\operatorname{conn} H}$$

is clear, because the spanning subgraphs of G are precisely the subgraphs of the form $(V, F, \varphi\mid_F)$ for some $F \subseteq E$.

Now, let $k \in \mathbb{N}$. We must prove that (# of proper k-colorings of G) $= \chi_G(k)$.

Let us substitute k for x in the equality

$$\chi_G = \sum_{F\subseteq E} (-1)^{|F|}\, x^{\operatorname{conn}(V,F,\varphi\mid_F)}.$$

We thus obtain

$$\begin{aligned}
&\chi_G(k) \\
&= \sum_{F \subseteq E} (-1)^{|F|} \underbrace{k^{\operatorname{conn}(V,F,\varphi|_F)}}_{\substack{= \sum\limits_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}} 1 \\ \text{(by Corollary 1.3.8)}}} = \sum_{F\subseteq E} (-1)^{|F|} \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}} 1 \\
&= \underbrace{\sum_{F\subseteq E} \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}}}_{= \sum\limits_{f:V\to\{1,2,\ldots,k\}} \sum\limits_{\substack{F\subseteq E;\\ F\subseteq E_f}}} \underbrace{(-1)^{|F|} 1}_{=(-1)^{|F|}} = \sum_{f:V\to\{1,2,\ldots,k\}} \underbrace{\sum_{\substack{F\subseteq E;\\ F\subseteq E_f}}}_{\substack{=\sum\limits_{F\subseteq E_f} \\ \text{(since } E_f\subseteq E)}} (-1)^{|F|} \\
&= \sum_{f:V\to\{1,2,\ldots,k\}} \sum_{F\subseteq E_f} (-1)^{|F|} = \sum_{f:V\to\{1,2,\ldots,k\}} \underbrace{\sum_{A\subseteq E_f} (-1)^{|A|}}_{\substack{=[E_f=\varnothing] \\ \text{(by Lemma 1.3.3,} \\ \text{applied to } P=E_f)}} \\
&\qquad \left(\begin{array}{c}\text{here, we have renamed the summation index } F \\ \text{in the inner sum as } A\end{array}\right) \\
&= \sum_{f:V\to\{1,2,\ldots,k\}} [E_f = \varnothing] \\
&= \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ E_f=\varnothing}} \underbrace{[E_f = \varnothing]}_{\substack{=1 \\ \text{(since } E_f=\varnothing \text{ is true)}}} + \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ \text{not } E_f=\varnothing}} \underbrace{[E_f = \varnothing]}_{\substack{=0 \\ \text{(since } E_f=\varnothing \text{ is false)}}} \\
&\qquad \left(\begin{array}{c}\text{since each } f : V \to \{1,2,\ldots,k\} \text{ either satisfies } E_f = \varnothing \\ \text{or does not}\end{array}\right) \\
&= \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ E_f=\varnothing}} 1 + \underbrace{\sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ \text{not } E_f=\varnothing}} 0}_{=0} = \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ E_f=\varnothing}} 1 \\
&= \left(\text{the number of all } f : V \to \{1,2,\ldots,k\} \text{ such that } E_f = \varnothing\right) \cdot 1 \\
&= \left(\text{the number of all } f : V \to \{1,2,\ldots,k\} \text{ such that } E_f = \varnothing\right) \\
&= (\text{the number of all } f : V \to \{1,2,\ldots,k\} \text{ such that the } k\text{-coloring } f \text{ is proper}) \\
&\qquad \left(\begin{array}{c}\text{since Proposition 1.3.6 shows that the condition ``} E_f = \varnothing \text{''} \\ \text{is equivalent to ``the } k\text{-coloring } f \text{ is proper''}\end{array}\right) \\
&= (\text{the number of all proper } k\text{-colorings}).
\end{aligned}$$

In other words, the number of proper k-colorings of G is $\chi_G(k)$. This completes the proof of Theorem 1.3.1. □

Definition 1.3.9. The polynomial χ_G in Theorem 1.3.1 is known as the **chromatic polynomial** of G.

Here are the chromatic polynomials of some graphs:

Proposition 1.3.10. Let $n \geq 1$ be an integer.

(a) For the path graph P_n with n vertices, we have

$$\chi_{P_n} = x(x-1)^{n-1}.$$

(b) More generally, for any tree T with n vertices, we have

$$\chi_T = x(x-1)^{n-1}.$$

(c) For the complete graph K_n with n vertices, we have

$$\chi_{K_n} = x(x-1)(x-2)\cdots(x-n+1).$$

(d) For the empty graph E_n with n vertices, we have

$$\chi_{E_n} = x^n.$$

(e) Assume that $n \geq 2$. For the cycle graph C_n with n vertices, we have

$$\chi_{C_n} = (x-1)^n + (-1)^n (x-1).$$

Proof sketch. **(c)** In order to prove that two polynomials with real coefficients are identical, it suffices to show that they agree on all nonnegative integers (this is an instance of the "principle of permanence of polynomial identities" that we already met in Lecture 21). Thus, in order to prove that $\chi_{K_n} = x(x-1)(x-2)\cdots(x-n+1)$, it suffices to show that $\chi_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1)$ for each $k \in \mathbb{N}$.

So let us do this. Fix $k \in \mathbb{N}$. Theorem 1.3.1 (applied to $G = K_n$) yields

$$(\text{\# of proper } k\text{-colorings of } K_n) = \chi_{K_n}(k). \tag{2}$$

Now, how many proper k-colorings does K_n have? We can construct such a proper k-coloring as follows:

- First, choose the color of the vertex 1. There are k options for this.
- Then, choose the color of the vertex 2. There are $k-1$ options for this, since it must differ from the color of 1.
- Then, choose the color of the vertex 3. There are $k-2$ options for this, since it must differ from the colors of 1 and of 2 (and the latter two colors are distinct, so we must subtract 2, not 1).

- And so on, until all n vertices are colored.

The total number of options to perform this construction is therefore $k(k-1)(k-2)\cdots(k-n+1)$. Hence,

$$(\#\text{ of proper } k\text{-colorings of } K_n) = k(k-1)(k-2)\cdots(k-n+1).$$

Comparing this with (2), we obtain $\chi_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1)$. As we already explained, this completes the proof of Proposition 1.3.10 **(c)**.

(d) This is similar to part **(c)**, but easier. We leave the proof to the reader. Alternatively, it follows easily from the definition of χ_{E_n}, since E_n has only one spanning subgraph (namely, E_n itself).

(b) (This is an outline; see [17s-mt2s, §0.6] for details.)

We proceed by induction on n. If $n = 1$, then this is easily checked by hand. If $n > 1$, then the tree T has at least one leaf (by Theorem 1.2.2 **(a)** in Lecture 14). Thus, we can fix a leaf ℓ of T. The graph $T \setminus \ell$ then is a tree (by Theorem 1.2.3 in Lecture 14) and has $n-1$ vertices, and therefore (by the induction hypothesis) its chromatic polynomial is $\chi_{T\setminus\ell} = x(x-1)^{n-2}$. However, for any given $k \in \mathbb{N}$, we can construct a proper k-coloring of T by first choosing a proper k-coloring of $T \setminus \ell$ and then choosing the color of the remaining leaf ℓ (there are $k-1$ choices for it, since it has to differ from the color of the unique neighbor of ℓ). Therefore, for each $k \in \mathbb{N}$, we have

$$(\#\text{ of proper } k\text{-colorings of } T) = (\#\text{ of proper } k\text{-colorings of } T\setminus\ell)\cdot(k-1).$$

In view of Theorem 1.3.1, this equality can be rewritten as

$$\chi_T(k) = \chi_{T\setminus\ell}(k)\cdot(k-1).$$

Since this holds for all $k \in \mathbb{N}$, we thus conclude that

$$\chi_T = \underbrace{\chi_{T\setminus\ell}}_{=x(x-1)^{n-2}}\cdot(x-1) = x(x-1)^{n-2}\cdot(x-1) = x(x-1)^{n-1}.$$

This completes the induction step.

Alternatively, Proposition 1.3.10 **(b)** can also be derived from the definition of χ_T, using the fact that every spanning subgraph H of T has no cycles and therefore satisfies $\operatorname{conn} H = n - |\operatorname{E}(H)|$ (by Corollary 1.1.8 in Lecture 13).

(a) This is a particular case of part **(b)**, since P_n is a tree with n vertices.

(e) There are different ways to prove this; see [LeeShi19] for four different proofs. The simplest one is probably by induction on n: Let $n \geq 2$. Fix $k \in \mathbb{N}$. A proper k-coloring of C_n is the same as a proper k-coloring of P_n that assigns different colors to the vertices 1 and n. Hence,

$$\begin{aligned}
&(\text{\# of proper } k\text{-colorings of } C_n) \\
&= (\text{\# of proper } k\text{-colorings of } P_n \text{ that assign different colors to 1 and } n) \\
&= \underbrace{(\text{\# of proper } k\text{-colorings of } P_n)}_{\substack{=k(k-1)^{n-1} \\ \text{(by part (a))}}} \\
&\qquad - \underbrace{(\text{\# of proper } k\text{-colorings of } P_n \text{ that assign the same color to 1 and } n)}_{\substack{=(\text{\# of proper } k\text{-colorings of } C_{n-1}) \\ \text{(why?)}}} \\
&= k\left(k-1\right)^{n-1} - (\text{\# of proper } k\text{-colorings of } C_{n-1}).
\end{aligned}$$

In view of Theorem 1.3.1, this equality can be rewritten as

$$\chi_{C_n}(k) = k\left(k-1\right)^{n-1} - \chi_{C_{n-1}}(k).$$

Since this holds for all $k \in \mathbb{N}$, we thus obtain

$$\chi_{C_n} = x\left(x-1\right)^{n-1} - \chi_{C_{n-1}}.$$

This is a recursion that is easily solved for χ_{C_n}, yielding the claim of part **(e)**. □

1.4. Vizing's theorem

So far we have been coloring the vertices of a graph. We can also color the edges:

Definition 1.4.1. Let $G = (V, E, \varphi)$ be a multigraph. Let $k \in \mathbb{N}$.

A k**-edge-coloring** of G means a map $f : E \to \{1, 2, \ldots, k\}$.

Such a k-edge-coloring f is called **proper** if no two distinct edges that have a common endpoint have the same color.

The most prominent fact about edge-colorings is the following theorem:

Theorem 1.4.2 (Vizing's theorem)**.** Let G be a simple graph with at least one vertex. Let

$$\alpha := \max\{\deg v \mid v \in V\}.$$

Then, G has a proper $(\alpha+1)$-edge-coloring.

Proof. See, e.g., [Schrij04] or various textbooks on graph theory.[9] □

Two remarks:

[9] Note that [Schrij04] uses some standard graph-theoretical notations: What we call α is denoted by $\Delta(G)$ in [Schrij04], whereas $\chi'(G)$ denotes the minimum $k \in \mathbb{N}$ for which G has a proper k-edge-coloring.

- The $\alpha + 1$ in Vizing's theorem cannot be improved in general (e.g., take G to be an odd-length cycle graph C_n).

- Vizing's theorem can be adapted to work for multigraphs instead of simple graphs. However, this requires replacing the $\alpha + 1$ by $\alpha + m$, where m is the maximum number of distinct mutually parallel edges in G (since otherwise, the multigraph $\left(K_3^{\text{bidir}}\right)^{\text{und}}$ would be a counterexample, as it has $\alpha = 4$ but has no proper 5-edge-coloring). For a proof of this, see [BerFou91, Corollary 2].

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