Math 530 Spring 2022, Lecture 21: Trees and colorings

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Trees and arborescences (cont'd)

1.1. A weighted Matrix-Tree Theorem

We have so far been **counting** arborescences. A natural generalization of counting is **weighted counting** – i.e., you assign a certain number (a "weight") to each arborescence (or whatever object you are interested in), and then you **sum** the weights of all arborescences (instead of merely counting them). This generalizes counting, because if all weights are 1, then you get the # of arborescences.

If you pick the weights to be completely random, then the sum won't usually be particularly interesting. However, some choices of weights lead to good behavior. Let us see what we get if we assign a weight to each **arc** of our digraph, and then define the weight of an arborescence to be the **product** of the weights of the arcs that appear in this arborescence.

Definition 1.1.1. Let $D = (V, A, \psi)$ be a multidigraph.

Let $\mathbb{K}$ be a commutative ring. Assume that an element $w_a \in \mathbb{K}$ is assigned to each arc $a \in A$. We call this w_a the **weight** of the arc a. (You can assume that $\mathbb{K} = \mathbb{R}$, so that the weights are just numbers.)

(a) For any two vertices $i, j \in V$, we let $a^w_{i,j}$ be the sum of the weights of all arcs of D that have source i and target j.

(b) For any vertex $i \in V$, we define the **weighted outdegree** $\deg^{+w} i$ of i to be the sum

$$\sum_{\substack{a \in A;\\ \text{the source of } a \text{ is } i}} w_a.$$

(c) If B is a subdigraph of D, then the **weight** $w(B)$ of B is defined to be the product $\prod_{a \text{ is an arc of } B} w_a$. This is the product of the weights of all arcs of B.

(d) Assume that $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. The **weighted Laplacian** of D (with respect to the weights w_a) is defined to be the $n \times n$-matrix $L^w \in \mathbb{K}^{n \times n}$ (note that the "w" here is a superscript, not an exponent) whose entries are given by

$$L^w_{i,j} = \left(\deg^{+w} i\right) \cdot [i = j] - a^w_{i,j} \qquad \text{for all } i, j \in V.$$

These definitions generalize analogous definitions in the "unweighted case". Indeed, if we take all the arc weights w_a to be 1, then the weighted outdegree $\deg^{+w} i$ of a vertex i becomes its usual outdegree $\deg i$, and the weighted Laplacian L^w becomes the usual Laplacian L. The weight $w(B)$ of a subdigraph B simply becomes 1 in this case.

We now can generalize the original MTT (= Matrix-Tree Theorem)[1] as follows:

Theorem 1.1.2 (weighted Matrix-Tree Theorem)**.** Let $D = (V, A, \psi)$ be a multidigraph.

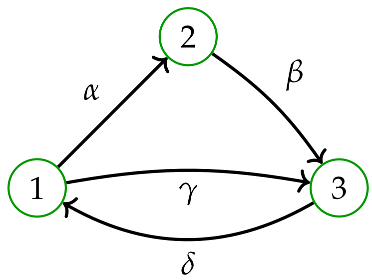
Let $\mathbb{K}$ be a commutative ring. Assume that an element $w_a \in \mathbb{K}$ is assigned to each arc $a \in A$. We call this w_a the **weight** of the arc a.

Assume that $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. Let L^w be the weighted Laplacian of D.

Let r be a vertex of D. Then,

$$\sum_{\substack{B \text{ is a spanning} \\ \text{arborescence} \\ \text{of } D \text{ rooted to } r}} w(B) = \det\left(L^w_{\sim r, \sim r}\right).$$

Example 1.1.3. Let D be the following multidigraph:

$D =$ 

, and let $r = 3$.

Then, D has two spanning arborescences rooted to r. One of the two has arcs α and β (and thus has weight $w_\alpha w_\beta$); the other has arcs γ and β (and thus has weight $w_\gamma w_\beta$). Hence,

$$\sum_{\substack{B \text{ is a spanning} \\ \text{arborescence} \\ \text{of } D \text{ rooted to } r}} w(B) = w_\alpha w_\beta + w_\gamma w_\beta, \tag{1}$$

The weighted Laplacian L^w is

$$L^w = \begin{pmatrix} w_\alpha + w_\gamma & -w_\alpha & -w_\gamma \\ 0 & w_\beta & -w_\beta \\ -w_\delta & 0 & w_\delta \end{pmatrix}$$

[1] To remind: The original MTT is Theorem 1.1.2 in Lecture 19.

(since, for example, $\deg^{+w} 1 = w_\alpha + w_\gamma$ and $a^w_{1,1} = 0$ and $a^w_{1,2} = w_\alpha$). Thus,

$$L^w_{\sim 3, \sim 3} = \begin{pmatrix} w_\alpha + w_\gamma & -w_\alpha \\ 0 & w_\beta \end{pmatrix} \qquad \text{and therefore}$$
$$\det \left(L^w_{\sim 3, \sim 3}\right) = (w_\alpha + w_\gamma)\, w_\beta = w_\alpha w_\beta + w_\gamma w_\beta.$$

The right hand side of this agrees with that of (1). This confirms the weighted MTT for our D and r.

As we already said, the weighted MTT generalizes the original MTT, because if we take all w_a's to be 1, we just recover the original MTT.

However, we can also go backwards: we can derive the weighted MTT from the original MTT. Let us do this.

First, we recall a standard result in algebra, known as the **principle of permanence of polynomial identities** or as the **polynomial identity trick** (it also goes under several other names). Here is one incarnation of this principle:

Theorem 1.1.4 (principle of permanence of polynomial identities)**.** Let $P(x_1, x_2, \ldots, x_m)$ and $Q(x_1, x_2, \ldots, x_m)$ be two polynomials with integer coefficients in several indeterminates $x_1, x_2, \ldots, x_m$. Assume that the equality

$$P(k_1, k_2, \ldots, k_m) = Q(k_1, k_2, \ldots, k_m) \tag{2}$$

holds for every m-tuple $(k_1, k_2, \ldots, k_m) \in \mathbb{N}^m$ of nonnegative integers. Then, $P(x_1, x_2, \ldots, x_m)$ and $Q(x_1, x_2, \ldots, x_m)$ are identical as polynomials (so that, in particular, the equality (2) holds not only for every $(k_1, k_2, \ldots, k_m) \in \mathbb{N}^m$, but also for every $(k_1, k_2, \ldots, k_m) \in \mathbb{C}^m$, and more generally, for every $(k_1, k_2, \ldots, k_m) \in \mathbb{K}^m$ where $\mathbb{K}$ is an arbitrary commutative ring).

Theorem 1.1.4 is often summarized as "in order to prove that two polynomials are equal, it suffices to show that they are equal on all nonnegative integer points" (where a "nonnegative integer point" means a point – i.e., a tuple of inputs – whose all entries are nonnegative integers). Even shorter, one says that "a polynomial identity (i.e., an equality between two polynomials) needs only to be checked on nonnegative integers". For example, if you can prove the equality

$$(x+y)^4 + (x-y)^4 = 2x^4 + 12x^2y^2 + 2y^4$$

for all nonnegative integers x and y, then you automatically conclude that this equality holds as a polynomial identity, and thus is true for any elements x and y of a commutative ring.

A typical application of Theorem 1.1.4 is to argue that a polynomial identity you have proved for all nonnegative integers must automatically hold for all inputs (because of Theorem 1.1.4). Some examples of such reasoning can be found in [19fco, §2.6.3 and §2.6.4]. A variant of Theorem 1.1.4 is [Conrad21, Theorem 2.6]; actually, the proof of [Conrad21, Theorem 2.6] can be trivially adapted to prove Theorem 1.1.4 (just replace "nonempty open set in $\mathbb{C}^k$" by "$\mathbb{N}^k$"). In truth, there is nothing special about nonnegative integers and the set $\mathbb{N}$; you could replace $\mathbb{N}$ by any infinite set of numbers (or even any sufficiently large set of numbers, where "sufficiently large" means "more than $\max\{\deg P, \deg Q\}$ many"). See [Alon02, Lemma 2.1] for a fairly general version of Theorem 1.1.4 that includes such cases[2].

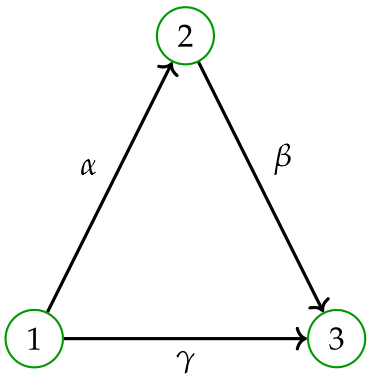
Proof of Theorem 1.1.2. The claim of Theorem 1.1.2 (for fixed D and r) is an equality between two polynomials in the arc weights w_a. (For instance, in Example 1.1.3, this equality is $w_\alpha w_\beta + w_\gamma w_\beta = \det \begin{pmatrix} w_\alpha + w_\gamma & -w_\alpha \\ 0 & w_\beta \end{pmatrix}$.)

Therefore, thanks to Theorem 1.1.4, it suffices to prove this equality in the case when all arc weights w_a are nonnegative integers. So let us WLOG assume that arc weights w_a are nonnegative integers.

Let us now replace each arc a of D by w_a many copies of the arc a (having the same source as a and the same target as a). The result is a new digraph D'. Here is an example:

Example 1.1.5. Let D be the digraph

$D =$



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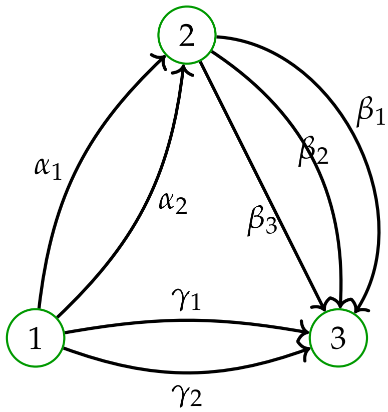
and let the arc weights be $w_\alpha = 2$ and $w_\beta = 3$ and $w_\gamma = 2$. Then, D' looks as

[2]To be precise, [Alon02, Lemma 2.1] is not concerned with two polynomials being identical, but rather with one polynomial being identically zero. But this is an equivalent question: Two polynomials P and Q are identical if and only if their difference $P - Q$ is identically zero.

follows:

$D' =$ 

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where α_1, α_2 are the two arcs obtained from α, and so on.

Now, recall that the digraph D' has the same vertices as D, but each arc a of D has turned into w_a arcs of D'. Thus, the weighted outdegree $\deg^{+w} i$ of a vertex i of D equals the (usual, i.e., non-weighted) outdegree $\deg^+ i$ of the same vertex i of D'. Hence, the weighted Laplacian L^w of D is the (usual, i.e., non-weighted) Laplacian of D'.

Recall again that the digraph D' has the same vertices as D, but each arc a of D has turned into w_a arcs of D'. Thus, each subdigraph B of D gives rise to $w(B)$ many subdigraphs of D' (because we can replace each arc a of B by any of the w_a many copies of this arc in D'). Moreover, this correspondence takes spanning arborescences to spanning arborescences[3], and we can obtain any spanning arborescence of D' in this way from exactly one B. Hence,

$$\sum_{\substack{B \text{ is a spanning} \\ \text{arborescence} \\ \text{of } D \text{ rooted to } r}} w(B) = \left(\# \text{ of spanning arborescences of } D' \text{ rooted to } r\right).$$

Thus, applying the original MTT to D' yields the weighted MTT for D (since the weighted Laplacian L^w of D is the (usual, i.e., non-weighted) Laplacian of D'). This completes the proof of Theorem 1.1.2.

[*Remark:* Alternatively, it is not hard to adapt our above proof of the original MTT to the weighted case.] □

The weighted MTT has some applications that wouldn't be obvious from the original MTT. Here is one:

Exercise 1. Let $n \geq 2$ be an integer, and let $d_1, d_2, \ldots, d_n$ be n positive integers. An **n-tree** shall mean a simple graph with vertex set $\{1, 2, \ldots, n\}$ that is a tree.

[3]More precisely: Let B be a subdigraph of D, and let B' be any of the $w(B)$ many subdigraphs of D' that are obtained from B through this correspondence. Then, B is a spanning arborescence of D rooted to r if and only if B' is a spanning arborescence of D' rooted to r.

We know from Cayley's theorem that there are n^{n-2} many n-trees. How many of these n-trees have the property that

$$\deg i = d_i \qquad \text{for each vertex } i\ ?$$

Solution. The n-trees are just the spanning trees of the complete graph K_n.

To incorporate the $\deg i = d_i$ condition into our count, we use a generating function. So let us **not** fix the numbers $d_1, d_2, \ldots, d_n$, but rather consider the polynomial

$$P(x_1, x_2, \ldots, x_n) := \sum_{T \text{ is a } n\text{-tree}} x_1^{\deg 1} x_2^{\deg 2} \cdots x_n^{\deg n} \tag{3}$$

in n indeterminates $x_1, x_2, \ldots, x_n$ (where $\deg i$ means the degree of i in T). Then, the $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$-coefficient of this polynomial $P(x_1, x_2, \ldots, x_n)$ is the # of n-trees T satisfying the property that

$$\deg i = d_i \qquad \text{for each vertex } i$$

(because each such n-tree T contributes a monomial $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ to the sum on the right hand side of (3), whereas any other n-tree T contributes a different monomial to this sum).

Let us assign to each edge ij of K_n the weight $w_{ij} := x_i x_j$. Then, the definition of $P(x_1, x_2, \ldots, x_n)$ rewrites as follows:

$$P(x_1, x_2, \ldots, x_n) = \sum_{T \text{ is an } n\text{-tree}} w(T),$$

where $w(T)$ denotes the product of the weights of all edges of T. (Indeed, for any subgraph T of K_n, the weight $w(T)$ equals $x_1^{\deg 1} x_2^{\deg 2} \cdots x_n^{\deg n}$, where $\deg i$ means the degree of i in T.)

We have assigned weights to the edges of the graph K_n; let us now assign the same weights to the arcs of the digraph K_n^{bidir}. That is, the two arcs $(ij, 1)$ and $(ij, 2)$ corresponding to an edge ij of K_n shall both have the weight

$$w_{(ij,1)} = w_{(ij,2)} = w_{ij} = x_i x_j. \tag{4}$$

As we are already used to, we can replace spanning trees of K_n by spanning arborescences of K_n^{bidir} rooted to 1, since the former are in bijection with the latter. Thus, we have

$$\begin{aligned} &(\text{\# of spanning trees of } K_n) \\ &= \left(\text{\# of spanning arborescences of } K_n^{\text{bidir}} \text{ rooted to } 1\right). \end{aligned}$$

Moreover, since this bijection preserves weights (because of (4)), we also have

$$\sum_{\substack{T \text{ is a spanning} \\ \text{tree of } K_n}} w(T) = \sum_{\substack{B \text{ is a spanning} \\ \text{arborescence of } K_n^{\text{bidir}} \\ \text{rooted to } 1}} w(B).$$

In other words,

$$\sum_{T \text{ is an } n\text{-tree}} w(T) = \sum_{\substack{B \text{ is a spanning} \\ \text{arborescence of } K_n^{\text{bidir}} \\ \text{rooted to } 1}} w(B)$$

(since the spanning trees of K_n are precisely the n-trees).

To compute the right hand side, we shall use the weighted Matrix-Tree Theorem. The weighted Laplacian of K_n^{bidir} (with the weights we have just defined) is the $n \times n$-matrix L^w with entries given by

$$\begin{aligned}
L^w_{i,j} &= \left(\deg^{+w} i\right) \cdot [i = j] - a^w_{i,j} \\
&= \begin{cases} \deg^{+w} i - a^w_{i,j}, & \text{if } i = j; \\ -a^w_{i,j}, & \text{if } i \neq j \end{cases} \\
&= \begin{cases} \deg^{+w} i, & \text{if } i = j; \\ -a^w_{i,j}, & \text{if } i \neq j \end{cases} \qquad \left(\begin{array}{c} \text{since } a^w_{i,j} = 0 \text{ when } i = j \\ \text{(because } K_n^{\text{bidir}} \text{ has no loops)} \end{array} \right) \\
&= \begin{cases} x_i \left(x_1 + x_2 + \cdots + x_n\right) - x_i x_j, & \text{if } i = j; \\ -x_i x_j, & \text{if } i \neq j \end{cases} \\
&\qquad \left(\begin{array}{c} \text{since } \deg^{+w} i = x_i x_1 + x_i x_2 + \cdots + x_i x_{i-1} + x_i x_{i+1} + \cdots + x_i x_n \\ = x_i \left(x_1 + x_2 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n\right) \\ = x_i \left(x_1 + x_2 + \cdots + x_n\right) - x_i x_i \\ = x_i \left(x_1 + x_2 + \cdots + x_n\right) - x_i x_j \text{ whenever } i = j, \\ \text{and since } a^w_{i,j} = x_i x_j \text{ whenever } i \neq j \end{array} \right) \\
&= [i = j]\, x_i \left(x_1 + x_2 + \cdots + x_n\right) - x_i x_j \\
&= x_i \left([i = j] \left(x_1 + x_2 + \cdots + x_n\right) - x_j\right).
\end{aligned}$$

We can find its minor $\det\left(L^w_{\sim 1, \sim 1}\right)$ without too much trouble (e.g., using row transformations similar to the ones we have done back in the proof of Cayley's theorem[4]); the result is

$$\det\left(L^w_{\sim 1, \sim 1}\right) = x_1 x_2 \cdots x_n \left(x_1 + x_2 + \cdots + x_n\right)^{n-2}.$$

[4]The first step, of course, is to factor an x_i out of the i-th row for each i.

Summarizing what we have done so far,

$$\begin{aligned}
P(x_1, x_2, \ldots, x_n) &= \sum_{T \text{ is an } n\text{-tree}} w(T) = \sum_{\substack{B \text{ is a spanning} \\ \text{arborescence of } K_n^{\text{bidir}} \\ \text{rooted to } 1}} w(B) \\
&= \det\left(L^w_{\sim 1, \sim 1}\right) \qquad \text{(by the weighted Matrix-Tree Theorem)} \\
&= x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}. \qquad (5)
\end{aligned}$$

As we recall, we are looking for the $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$-coefficient in this polynomial. From (5), we see that

$$\begin{aligned}
&\left(\text{the } x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}\text{-coefficient of } P(x_1, x_2, \ldots, x_n)\right) \\
&= \left(\text{the } x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}\text{-coefficient of } x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}\right) \\
&= \left(\text{the } x_1^{d_1 - 1} x_2^{d_2 - 1} \cdots x_n^{d_n - 1}\text{-coefficient of } (x_1 + x_2 + \cdots + x_n)^{n-2}\right)
\end{aligned}$$

(because when we multiply a polynomial by $x_1 x_2 \cdots x_n$, all the exponents in it get incremented by 1, so its coefficients just shift by a 1 in each exponent).

Now, how can we describe the coefficients of $(x_1 + x_2 + \cdots + x_n)^{n-2}$, or, more generally, of $(x_1 + x_2 + \cdots + x_n)^m$ for some $m \in \mathbb{N}$? These are the so-called **multinomial coefficients** (named in analogy to the binomial coefficients, which are their particular case for $n = 2$). Their definition is as follows: If $p_1, p_2, \ldots, p_n, q$ are nonnegative integers with $q = p_1 + p_2 + \cdots + p_n$, then the **multinomial coefficient** $\dbinom{q}{p_1, p_2, \ldots, p_n}$ is defined to be $\dfrac{q!}{p_1! p_2! \cdots p_n!}$. If $q \neq p_1 + p_2 + \cdots + p_n$, then it is defined to be 0 instead. In either case, this coefficient is easily seen to be an integer.[5] The **multinomial formula (aka multinomial theorem)** says that for each $k \in \mathbb{N}$, we have

$$\begin{aligned}
(x_1 + x_2 + \cdots + x_n)^k &= \sum_{\substack{i_1, i_2, \ldots, i_n \in \mathbb{N}; \\ i_1 + i_2 + \cdots + i_n = k}} \binom{k}{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \\
&= \sum_{i_1, i_2, \ldots, i_n \in \mathbb{N}} \binom{k}{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}
\end{aligned}$$

(it does not matter whether we restrict the sum by the condition $i_1 + i_2 + \cdots + i_n = k$ or not, since the coefficient $\dbinom{k}{i_1, i_2, \ldots, i_n}$ is defined to be 0 when this condition is violated anyway). Hence,

$$\left(\text{the } x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}\text{-coefficient of } (x_1 + x_2 + \cdots + x_n)^k\right) = \binom{k}{i_1, i_2, \ldots, i_n}$$

[5] See [23wd, Lecture 18, Section 4.12] for an introduction to multinomial coefficients.

for any $k \in \mathbb{N}$ and any $i_1, i_2, \ldots, i_n \in \mathbb{N}$. In particular,

$$\left(\text{the } x_1^{d_1-1} x_2^{d_2-1} \cdots x_n^{d_n-1}\text{-coefficient of } (x_1 + x_2 + \cdots + x_n)^{n-2}\right) = \binom{n-2}{d_1-1,\ d_2-1,\ \ldots,\ d_n-1}.$$

Summarizing, we find

$$\begin{aligned}
&\left(\text{the } x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}\text{-coefficient of } P(x_1, x_2, \ldots, x_n)\right) \\
&= \left(\text{the } x_1^{d_1-1} x_2^{d_2-1} \cdots x_n^{d_n-1}\text{-coefficient of } (x_1 + x_2 + \cdots + x_n)^{n-2}\right) \\
&= \binom{n-2}{d_1-1,\ d_2-1,\ \ldots,\ d_n-1}.
\end{aligned}$$

However, the $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$-coefficient of $P(x_1, x_2, \ldots, x_n)$ is the # of n-trees T satisfying the property that

$$\deg i = d_i \qquad \text{for each vertex } i$$

(as we have seen above). Thus, we have proved the following: □

Theorem 1.1.6 (refined Cayley's formula)**.** Let $n \geq 2$ be an integer, and let $d_1, d_2, \ldots, d_n$ be n positive integers. Then, the # of n-trees with the property that

$$\deg i = d_i \qquad \text{for each } i \in \{1, 2, \ldots, n\}$$

is the multinomial coefficient

$$\binom{n-2}{d_1-1,\ d_2-1,\ \ldots,\ d_n-1}.$$

The harmonic vector theorem for Laplacians (Theorem 1.2.1 in Lecture 20) also has a weighted version:

Theorem 1.1.7 (harmonic vector theorem for weighted Laplacians)**.** Let $D = (V, A, \psi)$ be a multidigraph, where $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. Let $\mathbb{K}$ be a commutative ring. Assume that an element $w_a \in \mathbb{K}$ is assigned to each arc $a \in A$. For each $r \in V$, let $\tau^w(D, r)$ be the sum of the weights of all the spanning arborescences of D rooted to r. Let f^w be the row vector $(\tau^w(D,1),\ \tau^w(D,2),\ \ldots,\ \tau^w(D,n))$. Let L^w be the weighted Laplacian of D. Then, $f^w L^w = 0$.

Proof. Similar to the unweighted case. □

2. Colorings

Now to something different: Let's color the vertices of a graph!

2.1. Definition

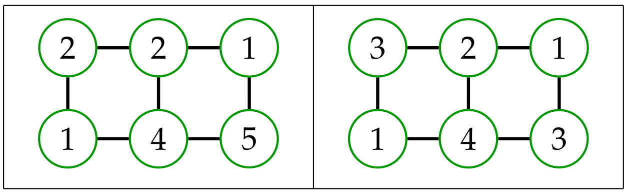
This is a serious course, so our colors are positive integers. Coloring the vertices thus means assigning a color (= a positive integer) to each vertex. Here are the details:

Definition 2.1.1. Let $G = (V, E, \varphi)$ be a multigraph. Let $k \in \mathbb{N}$.

(a) A **k-coloring** of G means a map $f : V \to \{1, 2, \ldots, k\}$. Given such a k-coloring f, we refer to the numbers $1, 2, \ldots, k$ as the **colors**, and we refer to each value $f(v)$ as the **color** of the vertex v in the k-coloring f.

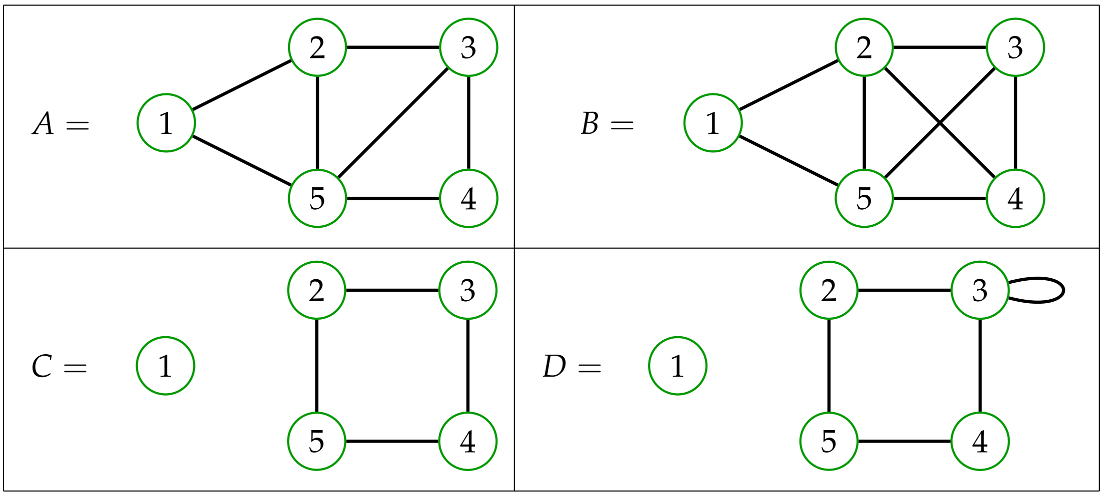
(b) A k-coloring f of G is said to be **proper** if no two adjacent vertices of G have the same color. (In other words, a k-coloring f of G is proper if there exists no edge of G whose endpoints u and v satisfy $f(u) = f(v)$.)

Example 2.1.2. Here are two 7-colorings of a graph:



(where the numbers on the nodes are not the vertices, but rather the colors of the vertices). The 7-coloring on the left (yes, it is a 7-coloring, even though it does not actually use the colors 3, 6 and 7) is not proper, because the two adjacent vertices on the top left have the same color. The 7-coloring on the right, however, is proper.

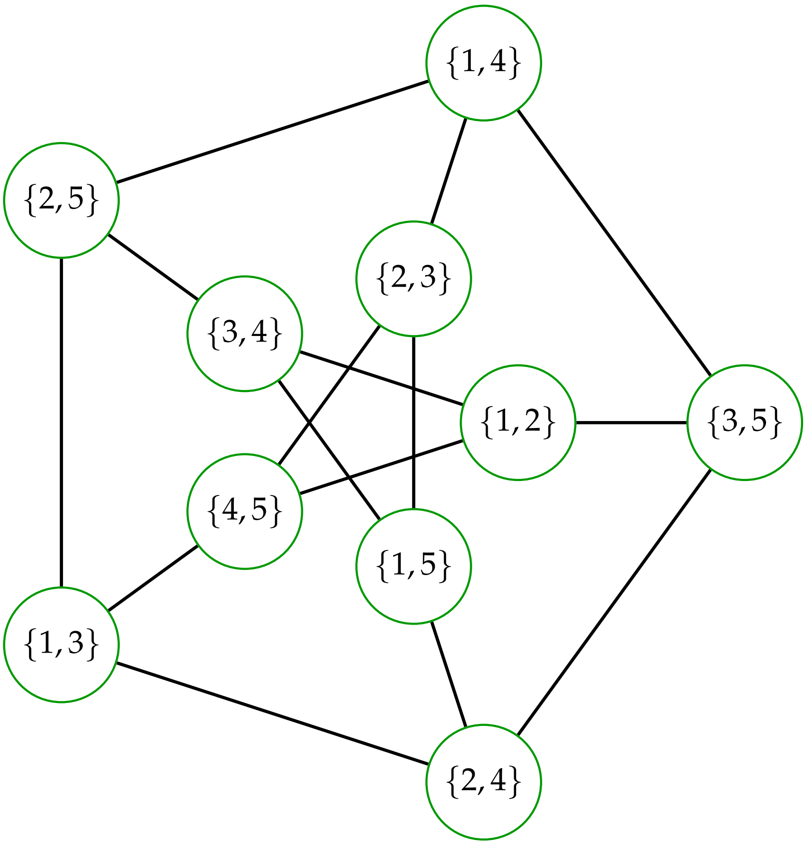
Example 2.1.3. Here is a bunch of graphs:



Which of them have proper 3-colorings?

- The graph A has a proper 3-coloring. For example, the map f that sends the vertices $1, 2, 3, 4, 5$ to the colors $1, 2, 1, 2, 3$ (respectively) is a proper 3-coloring.
- The graph B has no proper 3-coloring. Indeed, the four vertices $2, 3, 4, 5$ are mutually adjacent, so they would have to have 4 distinct colors in a proper k-coloring; but this is not possible unless $k \geq 4$.
- The graph C has a proper 3-coloring and even a proper 2-coloring (e.g., assigning color 1 to each odd vertex and color 2 to each even vertex).
- The graph D has no proper 3-coloring and, in fact, no proper k-coloring for any $k \in \mathbb{N}$. The reason is that the vertex 3 is adjacent to itself, but obviously has the same color as itself no matter what the k-coloring is. More generally, a graph with a loop cannot have a proper k-coloring for any $k \in \mathbb{N}$.

Example 2.1.4. Here is the Petersen graph:



I claim that it has a proper 3-coloring. Can you find it?

As we see, some graphs have proper 3-colorings, while others don't. Clearly, having 4 mutually adjacent vertices makes a proper 3-coloring impossible (indeed, by the pigeonhole principle, two of them must have the same color), but this is far from an "if and only if". The question of determining whether a given graph has a proper 3-coloring is NP-complete.

In contrast, the existence of proper 2-colorings is a much simpler question. The following is a nice criterion:

Theorem 2.1.5 (2-coloring equivalence theorem)**.** Let $G = (V, E, \varphi)$ be a multigraph. Then, the following three statements are equivalent:

- **Statement B1:** The graph G has a proper 2-coloring.
- **Statement B2:** The graph G has no cycles of odd length.
- **Statement B3:** The graph G has no circuits of odd length.

We will prove this next time.

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