Math 530 Spring 2022, Lecture 2: Degrees in simple graphs

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Simple graphs (cont'd)

1.1. Degrees

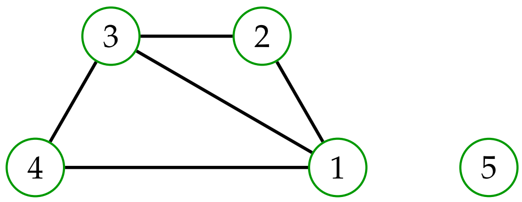
The **degree** of a vertex in a simple graph just counts how many edges contain this vertex:

Definition 1.1.1. Let $G = (V, E)$ be a simple graph. Let $v \in V$ be a vertex. Then, the **degree** of v (with respect to G) is defined to be

$$\begin{aligned} \deg v &:= (\text{the number of edges } e \in E \text{ that contain } v) \\ &= (\text{the number of neighbors of } v) \\ &= |\{u \in V \mid uv \in E\}| \\ &= |\{e \in E \mid v \in e\}| . \end{aligned}$$

(These equalities are pretty easy to check: Each edge $e \in E$ that contains v contains exactly one neighbor of v, and conversely, each neighbor of v belongs to exactly one edge that contains v. However, these equalities are specific to simple graphs, and won't hold any more once we move on to multigraphs.)

For example, in the graph



,

the vertices have degrees

$$\deg 1 = 3, \qquad \deg 2 = 2, \qquad \deg 3 = 3, \qquad \deg 4 = 2, \qquad \deg 5 = 0.$$

Here are some basic properties of degrees in simple graphs:

Proposition 1.1.2. Let G be a simple graph with n vertices. Let v be a vertex of G. Then,

$$\deg v \in \{0, 1, \ldots, n-1\} .$$

Proof. All neighbors of v belong to the $(n-1)$-element set $\operatorname{V}(G) \setminus \{v\}$. Thus, their number is $\leq n-1$. □

Proposition 1.1.3 (Euler 1736)**.** Let G be a simple graph. Then, the sum of the degrees of all vertices of G equals twice the number of edges of G. In other words,

$$\sum_{v \in \operatorname{V}(G)} \deg v = 2 \cdot |\operatorname{E}(G)|.$$

Proof. Write G as $G = (V, E)$; thus, $\operatorname{V}(G) = V$ and $\operatorname{E}(G) = E$.

Now, let N be the number of all pairs $(v, e) \in V \times E$ such that $v \in e$. We compute N in two different ways (this is called "double-counting"):

1. We can obtain N by computing, for each $v \in V$, the number of all $e \in E$ that satisfy $v \in e$, and then summing these numbers over all v. Since these numbers are just the degrees $\deg v$, the result will be $\sum_{v \in V} \deg v$.

2. On the other hand, we can obtain N by computing, for each $e \in E$, the number of all $v \in V$ that satisfy $v \in e$, and summing these numbers over all e. Since each $e \in E$ contains exactly 2 vertices $v \in V$, this result will be $\sum_{e \in E} 2 = |E| \cdot 2 = 2 \cdot |E|$.

Since these two results must be equal, we thus see that $\sum_{v \in V} \deg v = 2 \cdot |E|$. But that's the claim of Proposition 1.1.3. □

Corollary 1.1.4 (handshake lemma)**.** Let G be a simple graph. Then, the number of vertices v of G whose degree $\deg v$ is odd is even.

Proof. Proposition 1.1.3 yields that $\sum_{v \in \operatorname{V}(G)} \deg v = 2 \cdot |\operatorname{E}(G)|$. Hence, $\sum_{v \in \operatorname{V}(G)} \deg v$ is even. However, if a sum of integers is even, then it must have an even number of odd addends. Thus, the sum $\sum_{v \in \operatorname{V}(G)} \deg v$ must have an even number of odd addends. In other words, the number of vertices v of G whose degree $\deg v$ is odd is even. □

Corollary 1.1.4 is often stated as follows: In a group of people, the number of persons with an odd number of friends (in the group) is even. It is also known as the **handshake lemma**.

Here is another property of degrees in a simple graph:

Proposition 1.1.5. Let G be a simple graph with at least two vertices. Then, there exist two distinct vertices v and w of G that have the same degree.

Proof. Assume the contrary. So the degrees of all n vertices of G are distinct, where $n = |\operatorname{V}(G)|$.

In other words, the map

$$\begin{aligned} \deg : \operatorname{V}(G) &\to \{0, 1, \ldots, n-1\}, \\ v &\mapsto \deg v \end{aligned}$$

is injective. But this is a map between two finite sets of the same size (n). When such a map is injective, it has to be bijective (by the pigeonhole principle). Therefore, in particular, it takes both 0 and $n-1$ as values.

In other words, there are a vertex u with degree 0 and a vertex v with degree $n-1$. Are these two vertices adjacent or not? Yes because of $\deg v = n-1$; no because of $\deg u = 0$. Contradiction!

(Fine print: The two vertices u and v must be distinct, since $0 \neq n-1$. It is here that we are using the "at least two vertices" assumption!) □

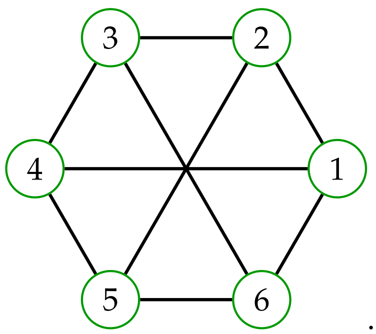
Here is an application of counting neighbors to proving a fact about graphs. This is known as **Mantel's theorem**:

Theorem 1.1.6 (Mantel's theorem)**.** Let G be a simple graph with n vertices and e edges. Assume that $e > n^2/4$. Then, G has a triangle (i.e., three distinct vertices that are pairwise adjacent).

Example 1.1.7. Let G be the graph (V, E), where

$$\begin{aligned} V &= \{1, 2, 3, 4, 5, 6\}; \\ E &= \{12,\ 23,\ 34,\ 45,\ 56,\ 61,\ 14,\ 25,\ 36\}. \end{aligned}$$

Here is a drawing:



This graph has no triangle (which, by the way, is easy to verify without checking all possibilities: just observe that every edge of G joins two vertices of different parity, but a triangle would necessarily have two vertices of equal parity). Thus, by the contrapositive of Mantel's theorem, it satisfies $e \leq n^2/4$ with $n = 6$ and $e = 9$. This is indeed true because $9 = 6^2/4$. But this also entails that if we add any further edge to G, then we obtain a triangle.

Proof of Mantel's theorem. We will prove the theorem by strong induction on n. Thus, we assume (as the induction hypothesis) that the theorem holds for all graphs with fewer than n vertices. We must now prove it for our graph G with n vertices. Let $V = \operatorname{V}(G)$ and $E = \operatorname{E}(G)$, so that $G = (V, E)$.

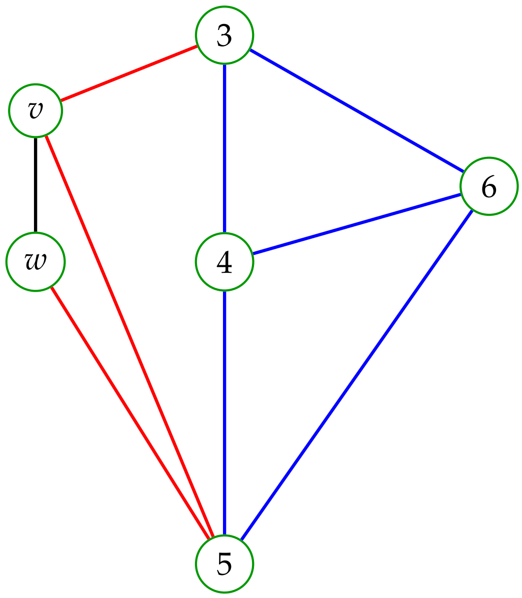
We must prove that G has a triangle. Assume the contrary. Thus, G has no triangle.

From $e > n^2/4 \geq 0$, we see that G has an edge. Pick any such edge, and call it vw. Thus, $v \neq w$.

Let us now color each edge of G with one of three colors, as follows:

- The edge vw is colored black.
- Each edge that contains exactly one of v and w is colored red.
- All other edges are colored blue.

The following picture shows an example of this coloring:



We now count the edges of each color:

- There is exactly 1 black edge – namely, vw.
- How many red edges can there be? I claim that there are at most $n-2$. Indeed, each vertex other than v and w is connected to at most one of v and w by a red edge, since otherwise it would form a triangle with v and w.
- How many blue edges can there be? The vertices other than v and w, along with the blue edges that join them, form a graph with $n-2$ vertices; this graph has no triangles (since G has no triangles). By the induction

hypothesis, however, if this graph had more than $(n-2)^2/4$ edges, then it would have a triangle. Thus, it has $\leq (n-2)^2/4$ edges. In other words, there are $\leq (n-2)^2/4$ blue edges.

In total, the number of edges is therefore

$$\leq 1 + (n-2) + (n-2)^2/4 = n^2/4.$$

In other words, $e \leq n^2/4$. This contradicts $e > n^2/4$. This is the contradiction we were looking for, so the induction is complete. □

Quick question: What about equality? Can a graph with n vertices and exactly $n^2/4$ edges have no triangles? Yes (for even n). Indeed, for any even n, we can take the graph

$$(\{1,2,\ldots,n\}, \ \{ij \mid i \not\equiv j \bmod 2\})$$

(keep in mind that ij means the 2-element set $\{i,j\}$ here, not the product $i \cdot j$). We can also do this for odd n, and obtain a graph with $\left(n^2-1\right)/4$ edges (which is as close to $n^2/4$ as we can get when n is odd – after all, the number of edges has to be an integer). So the bound in Mantel's theorem is optimal (as far as integers are concerned).

Mantel's theorem can be generalized:

Theorem 1.1.8 (Turan's theorem)**.** Let r be a positive integer. Let G be a simple graph with n vertices and e edges. Assume that

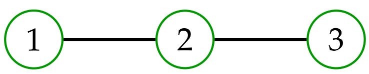
$$e > \frac{r-1}{r} \cdot \frac{n^2}{2}.$$

Then, there exist $r+1$ distinct vertices of G that are mutually adjacent.

Mantel's theorem is the particular case for $r = 2$. We will see a proof of Turan's theorem later (Theorem 1.3.1 in Lecture 23).

1.2. Graph isomorphism

Two graphs can be distinct and yet "the same up to the names of their vertices": for instance,



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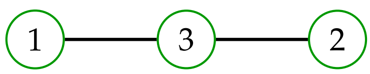

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Let us formalize this:

Definition 1.2.1. Let G and H be two simple graphs.

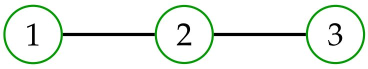
(a) A **graph isomorphism** (or **isomorphism**) from G to H means a bijection $\phi : \operatorname{V}(G) \to \operatorname{V}(H)$ that "preserves edges", i.e., that has the following property: For any two vertices u and v of G, we have

$$(uv \in \operatorname{E}(G)) \iff (\phi(u)\,\phi(v) \in \operatorname{E}(H)).$$

(b) We say that G and H are **isomorphic** (this is written $G \cong H$) if there exists a graph isomorphism from G to H.

Here are two examples:

- The two graphs



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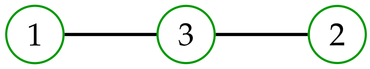
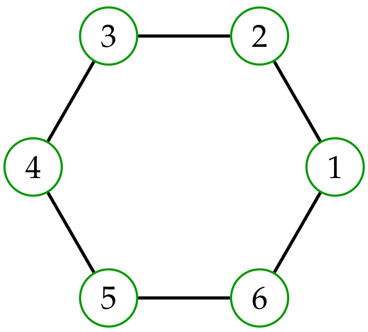
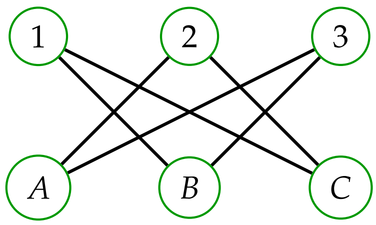


are isomorphic, because the bijection between their vertex sets that sends $1, 2, 3$ to $1, 3, 2$ is an isomorphism. Another isomorphism between the same two graphs sends $1, 2, 3$ to $2, 3, 1$.

- The two graphs



and



are isomorphic, because the bijection between their vertex sets that sends $1, 2, 3, 4, 5, 6$ to $1, B, 3, A, 2, C$ is an isomorphism.

Here are some basic properties of isomorphisms (the proofs are straightforward):

Proposition 1.2.2. Let G and H be two graphs. The inverse of a graph isomorphism ϕ from G to H is a graph isomorphism from H to G.

Proposition 1.2.3. Let G, H and I be three graphs. If ϕ is a graph isomorphism from G to H, and ψ is a graph isomorphism from H to I, then $\psi \circ \phi$ is a graph isomorphism from G to I.

As a consequence of these two propositions, it is easy to see that the relation $\cong$ (on the class of all graphs) is an equivalence relation.

Graph isomorphisms preserve all "intrinsic" properties of a graph. For example:

Proposition 1.2.4. Let G and H be two simple graphs, and ϕ a graph isomorphism from G to H. Then:

(a) For every $v \in \operatorname{V}(G)$, we have $\deg_G v = \deg_H (\phi(v))$. Here, $\deg_G v$ means the degree of v as a vertex of G, whereas $\deg_H (\phi(v))$ means the degree of $\phi(v)$ as a vertex of H.

(b) We have $|\operatorname{E}(H)| = |\operatorname{E}(G)|$.

(c) We have $|\operatorname{V}(H)| = |\operatorname{V}(G)|$.

One use of graph isomorphisms is to relabel the vertices of a graph. For example, we can relabel the vertices of an n-vertex graph as $1, 2, \ldots, n$, or as any other n distinct objects:

Proposition 1.2.5. Let G be a simple graph. Let S be a finite set such that $|S| = |\operatorname{V}(G)|$. Then, there exists a simple graph H that is isomorphic to G and has vertex set $\operatorname{V}(H) = S$.

Proof. Straightforward. □