Math 530 Spring 2022, Lecture 19: Trees

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Trees and arborescences (cont'd)

1.1. The matrix-tree theorem (cont'd)

Last time, we defined the Laplacian (matrix) of a multidigraph. Let us recall its definition:

Definition 1.1.1. Let $D = (V, A, \psi)$ be a multidigraph. Assume that $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

For any $i, j \in V$, we let $a_{i,j}$ be the # of arcs of D that have source i and target j.

The **Laplacian** of D is defined to be the $n \times n$-matrix $L \in \mathbb{Z}^{n \times n}$ whose entries are given by

$$L_{i,j} = \left(\deg^+ i\right) \cdot \underbrace{[i = j]}_{\substack{\text{This is also} \\ \text{known as } \delta_{i,j}}} - a_{i,j} \qquad \text{for all } i, j \in V.$$

In other words, it is the matrix

$$L = \begin{pmatrix} \deg^+ 1 - a_{1,1} & -a_{1,2} & \cdots & -a_{1,n} \\ -a_{2,1} & \deg^+ 2 - a_{2,2} & \cdots & -a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & -a_{n,2} & \cdots & \deg^+ n - a_{n,n} \end{pmatrix}.$$

We showed that $\det L = 0$ whenever $n > 0$. We stated (without proof) the following crucial result:

Theorem 1.1.2 (Matrix-Tree Theorem)**.** Let $D = (V, A, \psi)$ be a multidigraph. Assume that $V = \{1, 2, \ldots, n\}$ for some positive integer n.

Let L be the Laplacian of D. Let r be a vertex of D. Then,

$$(\text{# of spanning arborescences of } D \text{ rooted to } r) = \det\left(L_{\sim r, \sim r}\right).$$

We shall now prove this theorem, guided by the following battle plan:

1. First, we will prove it in the case when each vertex $v \in V \setminus \{r\}$ has outdegree 1. In this case, after removing all arcs with source r from D (these

arcs do not matter, since neither the submatrix $L_{\sim r,\sim r}$ nor the spanning arborescences rooted to r depend on them), we have essentially two options (subcases): either D is itself an arborescence or D has a cycle.

2. Then, we will prove the matrix-tree theorem in the slightly more general case when each $v \in V \setminus \{r\}$ has outdegree ≤ 1. This is easy, since a vertex $v \in V \setminus \{r\}$ having outdegree 0 trivializes the theorem.

3. Finally, we will prove the theorem in the general case. This is done by strong induction on the number of arcs of D. Every time you have a vertex $v \in V \setminus \{r\}$ with outdegree > 1, you can pick such a vertex and color the outgoing arcs from it red and blue in such a way that each color is used at least once. Then, you can consider the subdigraph of D obtained by removing all blue arcs (call it D^{red}) and the subdigraph of D obtained by removing all red arcs (call it D^{blue}). You can then apply the induction hypothesis to D^{red} and to D^{blue} (since each of these two subdigraphs has fewer arcs than D), and add the results together. The good news is that both the # of spanning arborescences rooted to r and the determinant $\det(L_{\sim r,\sim r})$ "behave additively" (we will soon see what this means).

So let us begin with Step 1. We first study a very special case:

Lemma 1.1.3. Let $D = (V, A, \psi)$ be a multidigraph. Let r be a vertex of D. Assume that D has no cycles. Assume moreover that D has no arcs with source r. Assume furthermore that each vertex $v \in V \setminus \{r\}$ has outdegree 1. Then:

(a) The digraph D has a unique spanning arborescence rooted to r.

(b) Assume that $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. Let L be the Laplacian of D. Then, $\det(L_{\sim r,\sim r}) = 1$.

Proof. **(a)** Lemma 1.2.10 in Lecture 18 shows that the digraph D itself is an arborescence rooted to r.

As a consequence, D itself is a spanning arborescence of D rooted to r.

Therefore, $|A| = |V| - 1$ (by Statement A'2 in the Dual Arborescence Equivalence Theorem[1]). Hence, D has no spanning arborescences other than itself (because the condition $|A| = |V| - 1$ would get destroyed as soon as we remove an arc). So the only spanning arborescence of D rooted to r is D itself. This proves Lemma 1.1.3 **(a)**.

(b) We WLOG assume that $r = n$ (otherwise, we can rename the vertices $r, r+1, r+2, \ldots, n$ as $n, r, r+1, \ldots, n-1$, so that the matrix $L_{\sim r,\sim r}$ becomes $L_{\sim n,\sim n}$).

[1] or by the fact that $|A|$ is the sum of the outdegrees of all vertices of D

Let D' be the digraph D with a loop added at each vertex – i.e., the multidigraph obtained from D by adding n extra arcs $\ell_1, \ell_2, \ldots, \ell_n$ and letting each arc ℓ_i have source i and target i.

Let S_{n-1} denote the group of permutations of the set

$$\{1,2,\ldots,n-1\} = \underbrace{\{1,2,\ldots,n\}}_{=V} \setminus \left\{ \underbrace{n}_{=r} \right\} = V \setminus \{r\}.$$

Now, from $r = n$, we have

$$\det(L_{\sim r,\sim r}) = \det(L_{\sim n,\sim n}) = \sum_{\sigma \in S_{n-1}} \operatorname{sign} \sigma \cdot \prod_{i=1}^{n-1} L_{i,\sigma(i)} \tag{1}$$

(by the Leibniz formula for the determinant). We shall now study the addends in the sum on the right hand side of this equality. Specifically, we will show that the only addend whose product $\prod_{i=1}^{n-1} L_{i,\sigma(i)}$ is nonzero is the addend for $\sigma = \operatorname{id}$.

Indeed, let $\sigma \in S_{n-1}$ be a permutation such that the product $\prod_{i=1}^{n-1} L_{i,\sigma(i)}$ is nonzero. We shall prove that $\sigma = \operatorname{id}$.

Consider an arbitrary $v \in \{1,2,\ldots,n-1\}$. Then, $L_{v,\sigma(v)} \neq 0$ (because $L_{v,\sigma(v)}$ is a factor in the product $\prod_{i=1}^{n-1} L_{i,\sigma(i)}$, which is nonzero). However, the definition of L yields $L_{v,\sigma(v)} = (\deg^+ v) \cdot [v = \sigma(v)] - a_{v,\sigma(v)}$. Thus,

$$(\deg^+ v) \cdot [v = \sigma(v)] - a_{v,\sigma(v)} = L_{v,\sigma(v)} \neq 0.$$

Hence, at least one of the numbers $[v = \sigma(v)]$ and $a_{v,\sigma(v)}$ is nonzero. In other words, we have $v = \sigma(v)$ (this is what it means for $[v = \sigma(v)]$ to be nonzero) or the digraph D has an arc with source v and target $\sigma(v)$ (because this is what it means for $a_{v,\sigma(v)}$ to be nonzero). In either case, the digraph D' has an arc with source v and target $\sigma(v)$ (because if $v = \sigma(v)$, then one of the loops we added to D does the trick). We can apply the same argument to $\sigma(v)$ instead of v, and obtain an arc with source $\sigma(v)$ and target $\sigma(\sigma(v))$. Similarly, we obtain an arc with source $\sigma(\sigma(v))$ and target $\sigma(\sigma(\sigma(v)))$. We can continue this reasoning indefinitely. By continuing it for n steps, we obtain a walk

$$\left(v, *, \sigma(v), *, \sigma^2(v), *, \sigma^3(v), \ldots, *, \sigma^n(v)\right)$$

in the digraph D', where each asterisk means an arc (we don't care about what these arcs are, so we are not giving them names). This walk cannot be a path (since it has $n+1$ vertices, but D' has only n vertices); thus, it must contain a cycle (by Proposition 1.2.9 in Lecture 10). All arcs of this cycle must be loops (because otherwise, we could remove the loops from this cycle and obtain a cycle of D, but we know that D has no cycles). In particular, its first arc is a loop. Thus, our above walk $\left(v, *, \sigma\left(v\right), *, \sigma^2\left(v\right), *, \sigma^3\left(v\right), \ldots, *, \sigma^n\left(v\right)\right)$ contains a loop (since the arcs of the cycle come from this walk). In other words, we have $\sigma^i\left(v\right) = \sigma^{i+1}\left(v\right)$ for some $i \in \{0, 1, \ldots, n-1\}$. Since σ is injective, we can apply σ^{-i} to both sides of this equality, and conclude that $v = \sigma\left(v\right)$. In other words, $\sigma\left(v\right) = v$.

Forget that we fixed v. We thus have shown that $\sigma\left(v\right) = v$ for each $v \in \{1, 2, \ldots, n-1\}$. In other words, $\sigma = \mathrm{id}$.

Forget that we fixed σ. We thus have proved that $\sigma = \mathrm{id}$ for each permutation $\sigma \in S_{n-1}$ for which the product $\prod\limits_{i=1}^{n-1} L_{i,\sigma(i)}$ is nonzero. In other words, the only permutation $\sigma \in S_{n-1}$ for which the product $\prod\limits_{i=1}^{n-1} L_{i,\sigma(i)}$ is nonzero is the permutation id.

Thus, the only nonzero addend on the right hand side of (1) is the addend corresponding to $\sigma = \mathrm{id}$. Hence, (1) can be simplified as follows:

$$\det\left(L_{\sim n,\sim n}\right) = \underbrace{\mathrm{sign}\left(\mathrm{id}\right)}_{=1} \cdot \prod_{i=1}^{n-1} L_{i,\mathrm{id}(i)} = \prod_{i=1}^{n-1} L_{i,\mathrm{id}(i)}.$$

Since each $i \in \{1, 2, \ldots, n-1\}$ satisfies

$$\begin{aligned} L_{i,\mathrm{id}(i)} = L_{i,i} &= \underbrace{\left(\deg^+ i\right)}_{\substack{=1 \\ \text{(since } i \text{ has outdegree 1} \\ \text{(because each vertex } v \in V \setminus \{r\} \text{ has} \\ \text{outdegree 1, and we can apply this} \\ \text{to } v=i \text{ since } i \in \{1,2,\ldots,n-1\} = V\setminus\{r\}))}} \cdot \underbrace{[i = i]}_{=1} - \underbrace{a_{i,i}}_{\substack{=0 \\ \text{(since } D \text{ has no cycles} \\ \text{and thus cannot have} \\ \text{a loop with source } i)}} \\ &\qquad \text{(by the definition of } L\text{)} \\ &= 1 \cdot 1 - 0 = 1, \end{aligned}$$

this can be simplified to $\det\left(L_{\sim n,\sim n}\right) = \prod\limits_{i=1}^{n-1} 1 = 1$. This proves Lemma 1.1.3 **(b)**. ☐

Next, we drop the "no cycles" condition:

> **Lemma 1.1.4.** Let $D = (V, A, \psi)$ be a multidigraph. Let r be a vertex of D. Assume that each vertex $v \in V \setminus \{r\}$ has outdegree 1. Then, the MTT holds for these D and r. (Here and in the following, "**MTT**" is short for "Matrix-Tree Theorem", i.e., for Theorem 1.1.2.)

Proof. First of all, we note that an arc with source r cannot appear in any spanning arborescence of D rooted to r (since any such arborescence satisfies

$\deg^+ r = 0$, according to Statement A'6 in the Dual Arborescence Equivalence Theorem). Furthermore, the arcs with source r do not affect the matrix $L_{\sim r,\sim r}$, since they only appear in the r-th row of the matrix L (but this r-th row is removed in $L_{\sim r,\sim r}$).

Hence, any arc with source r can be removed from D without disturbing anything we currently care about. Thus, we WLOG assume that D has no arcs with source r (else, we can just remove them from D).

We WLOG assume that $r = n$ (otherwise, we can rename the vertices $r, r+1, r+2, \ldots, n$ as $n, r, r+1, \ldots, n-1$, so that the matrix $L_{\sim r,\sim r}$ becomes $L_{\sim n,\sim n}$).

We are in one of the following two cases:

Case 1: The digraph D has a cycle.

Case 2: The digraph D has no cycles.

Consider Case 1. In this case, D has a cycle $\mathbf{v} = (v_1, *, v_2, *, \ldots, *, v_m)$ (where we again are putting asterisks in place of the arcs). This cycle cannot contain r (since D has no arcs with source r). Thus, all its vertices $v_1, v_2, \ldots, v_m$ belong to $V \setminus \{r\}$. Hence, for each $i \in \{1, 2, \ldots, m-1\}$, the vertex v_i has outdegree 1 (since we assumed that each vertex $v \in V \setminus \{r\}$ has outdegree 1). Consequently, for each $i \in \{1, 2, \ldots, m-1\}$, the only arc of D that has source v_i is the arc that follows v_i on the cycle $\mathbf{v}$. Therefore, in the matrix L, the v_i-th row has a 1 in the v_i-th position (because $\deg^+(v_i) = 1$), a -1 in the v_{i+1}-th position (since the arc that follows v_i on the cycle $\mathbf{v}$ has source v_i and target v_{i+1}), and 0s in all other positions. Since $r = n$, the same must then be true for the matrix $L_{\sim r,\sim r}$: That is, the v_i-th row of the matrix $L_{\sim r,\sim r}$ has a 1 in the v_i-th position, a -1 in the v_{i+1}-th position, and 0s in all other positions. Thus, the sum of the v_1-th, v_2-th, …, v_{m-1}-th rows of $L_{\sim r,\sim r}$ is the zero vector (since the 1s and the -1s just cancel out)[2].[3]

So we have found a nonempty set of rows of $L_{\sim r,\sim r}$ whose sum is the zero vector. This yields that the matrix $L_{\sim r,\sim r}$ is singular (by basic properties of

[2] Namely, the -1 in the v_{i+1}-th position of the v_i-th row gets cancelled by the 1 in the v_{i+1}-th position of the v_{i+1}-th row. (We are using the fact that $v_m = v_1$ here.)

[3] Let me illustrate this on a representative example: Assume that the numbers $v_1, v_2, \ldots, v_{m-1}, v_m$ are $1, 2, \ldots, m-1, 1$ (respectively). Then, the first $m-1$ rows of L look as follows:

$$\begin{array}{cccccc} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & 1 & -1 & & \\ & & & \ddots & \ddots & \\ & & & & 1 & -1 \\ -1 & & & & & 1 \end{array}$$

(where all the missing entries are zeroes). Thus, the sum of these $m-1$ rows is the zero vector. The same is therefore true of the matrix $L_{\sim r,\sim r}$ (since the first $m-1$ rows of the latter matrix are just the first $m-1$ rows of L, with their r-th entries removed).

The general case is essentially the same as this example; the only difference is that the relevant rows are in other positions.

determinants[4]), so its determinant is $\det(L_{\sim r,\sim r}) = 0$. On the other hand, the digraph D has no spanning arborescence (because, in order to get a spanning arborescence of D, we would have to remove at least one arc of our cycle $\mathbf{v}$ (since an arborescence cannot have a cycle); but then, the source of this arc would have outdegree 0, and thus we could no longer find a path from this source to r, so we would not obtain a spanning arborescence). In other words,

$$(\text{\# of spanning arborescences of } D \text{ rooted to } r) = 0.$$

Comparing this with $\det(L_{\sim r,\sim r}) = 0$, we conclude that the MTT holds in this case (since it claims that $0 = 0$). Thus, Case 1 is done.

Next, we consider Case 2. In this case, D has no cycles. Then, $\det(L_{\sim r,\sim r}) = 1$ (by Lemma 1.1.3 **(b)**) and

$$(\text{\# of spanning arborescences of } D \text{ rooted to } r) = 1 \qquad (\text{by Lemma 1.1.3 } \textbf{(a)}).$$

Thus, the MTT boils down to $1 = 1$, which is again true.

So Lemma 1.1.4 is proved. □

Next, we venture into a mildly greater generality:

Lemma 1.1.5. Let $D = (V, A, \psi)$ be a multidigraph. Let r be a vertex of D. Assume that each vertex $v \in V \setminus \{r\}$ has outdegree ≤ 1. Then, the MTT (= Matrix-Tree Theorem) holds for these D and r.

Proof. If each vertex $v \in V \setminus \{r\}$ has outdegree 1, then this is true by Lemma 1.1.4.

Thus, we WLOG assume that this is not the case. Hence, some vertex $v \in V \setminus \{r\}$ has outdegree $\neq 1$. Consider this v. The outdegree of v is $\neq 1$, but also ≤ 1 (by the hypothesis of the lemma). Hence, this outdegree must be 0. That is, there is no arc with source v.

WLOG assume that $r = n$ (otherwise, rename the vertices $r, r+1, r+2, \ldots, n$ as $n, r, r+1, \ldots, n-1$, so that the matrix $L_{\sim r,\sim r}$ becomes $L_{\sim n,\sim n}$).

We have $v \neq r$. Hence, the digraph D has no path from v to r (since any such path would include an arc with source v, but there is no arc with source v).

Therefore, D has no spanning arborescence rooted to r (because any such spanning arborescence would have to have a path from v to r). In other words,

$$(\text{\# of spanning arborescences of } D \text{ rooted to } r) = 0.$$

[4]Specifically, we are using the following fact: "Let M be a square matrix. If there is a certain nonempty set of rows of M whose sum is the zero vector, then the matrix M is singular.".

To prove this fact, we let S be this nonempty set. Choose one row from this set, and call it the **chosen row**. Now, add all the other rows from this set to this one chosen row. This operation does not change the determinant of M (since the determinant of a matrix is unchanged when we add one row to another), but the resulting matrix has a zero row (namely, the chosen row) and thus has determinant 0. Hence, the original matrix M must have had determinant 0 as well. In other words, M was singular, qed.

Also, $\det\left(L_{\sim r,\sim r}\right) = 0$ (since the v-th row of the matrix $L_{\sim r,\sim r}$ is 0 (because there is no arc with source v)). So the MTT boils down to $0 = 0$ again, and thus Lemma 1.1.5 is proved. □

We are now ready to prove the MTT in the general case:

Proof of Theorem 1.1.2. First, we introduce a notation:

> Let M and N be two $n \times n$-matrices that agree in all but one row. That is, there exists some $j \in \{1, 2, \ldots, n\}$ such that for each $i \neq j$, we have
> $$(\text{the } i\text{-th row of } M) = (\text{the } i\text{-th row of } N).$$
> Then, we write $M \overset{j}{\equiv} N$, and we let $M \overset{j}{+} N$ be the $n \times n$-matrix that is obtained from M by adding the j-th row of N to the j-th row of M (while leaving all remaining rows unchanged).

For example, if $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $N = \begin{pmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{pmatrix}$, then $M \overset{2}{\equiv} N$ and

$$M \overset{2}{+} N = \begin{pmatrix} a & b & c \\ d + d' & e + e' & f + f' \\ g & h & i \end{pmatrix}.$$

A well-known property of determinants (the **multilinearity of the determinant**) says that if M and N are two $n \times n$-matrices and $j \in \{1, 2, \ldots, n\}$ is a number such that $M \overset{j}{\equiv} N$, then

$$\det\left(M \overset{j}{+} N\right) = \det M + \det N.$$

Now, let us prove the MTT. We proceed by strong induction on the # of arcs of D.

Induction step: Let $m \in \mathbb{N}$. Assume (as the induction hypothesis) that the MTT holds for all digraphs D that have $< m$ arcs. We must now prove it for our digraph D with m arcs.

WLOG assume that $r = n$ (otherwise, rename the vertices $r, r+1, r+2, \ldots, n$ as $n, r, r+1, \ldots, n-1$, so that the matrix $L_{\sim r,\sim r}$ becomes $L_{\sim n,\sim n}$).

If each vertex $v \in V \setminus \{r\}$ has outdegree ≤ 1, then the MTT holds by Lemma 1.1.5. Thus, we WLOG assume that some vertex $v \in V \setminus \{r\}$ has outdegree > 1. Pick such a vertex v. We color each arc with source v either red or blue, making sure that at least one arc is red and at least one arc is blue. (We can do this, since v has outdegree > 1.) All arcs that do not have source v remain uncolored.

Now, let D^{red} be the subdigraph obtained from D by removing all blue arcs. Then, D^{red} has fewer arcs than D. In other words, D^{red} has $< m$ arcs. Hence, the induction hypothesis yields that the MTT holds for D^{red}. That is, we have

$$\left(\# \text{ of spanning arborescences of } D^{\text{red}} \text{ rooted to } r\right) = \det\left(L^{\text{red}}_{\sim r,\sim r}\right),$$

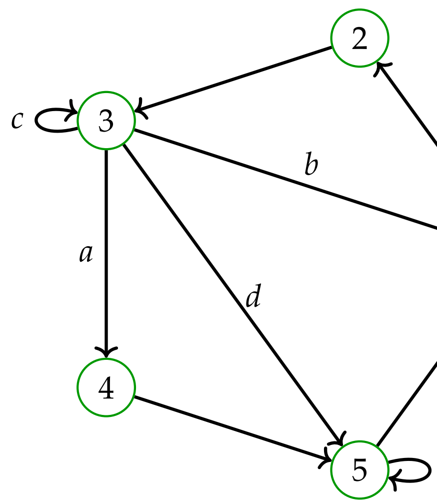
where L^{red} means the Laplacian of D^{red}.

Likewise, let D^{blue} be the subdigraph obtained from D by removing all red arcs. Then, D^{blue} has fewer arcs than D. Hence, the induction hypothesis yields that the MTT holds for D^{blue}. That is,

$$\left(\# \text{ of spanning arborescences of } D^{\text{blue}} \text{ rooted to } r\right) = \det\left(L^{\text{blue}}_{\sim r,\sim r}\right),$$

where L^{blue} means the Laplacian of D^{blue}.

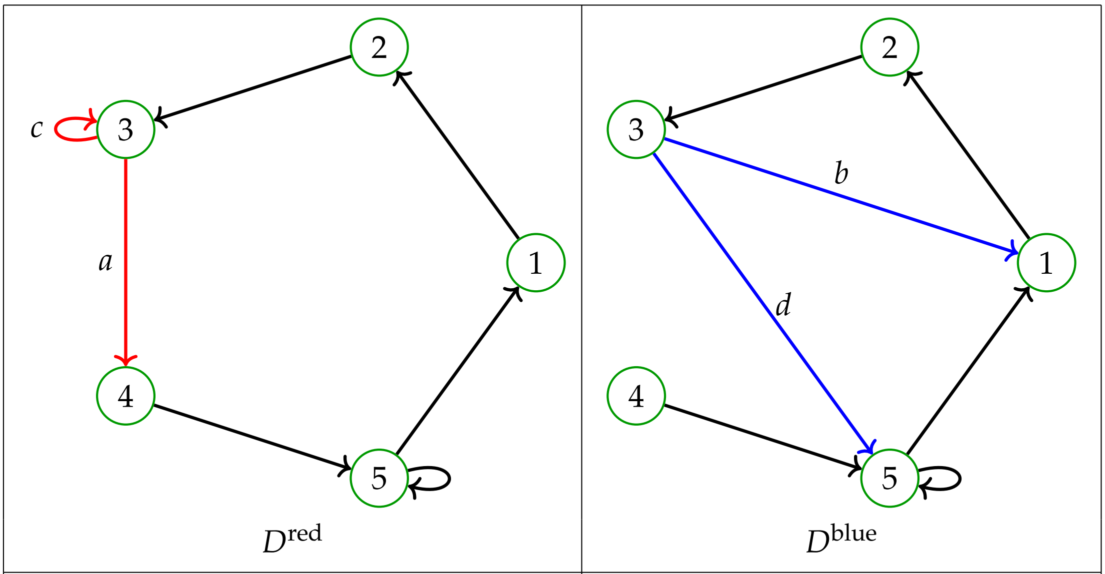
Example 1.1.6. Let D be the multidigraph



with $r = 1$. Its Laplacian is

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us pick $v = 3$ (this is a vertex with outdegree > 1), and let us color the arcs a and c red and the arcs b and d blue (various other options are possible). Then, D^{red} and D^{blue} look as follows (along with their Laplacians L^{red} and L^{blue}):



$$L^{\text{red}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad L^{\text{blue}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, the digraphs D, D^{blue} and D^{red} differ only in the arcs with source v, and as far as the latter arcs are concerned, the arcs of D are divided between D^{blue} and D^{red}. Hence, by the definition of the Laplacian, we have

$$L^{\text{red}} \overset{v}{\equiv} L^{\text{blue}} \qquad \text{and} \qquad L^{\text{red}} \overset{v}{+} L^{\text{blue}} = L.$$

Thus,

$$L^{\text{red}}_{\sim r,\sim r} \overset{v}{\equiv} L^{\text{blue}}_{\sim r,\sim r} \qquad \text{and} \qquad L^{\text{red}}_{\sim r,\sim r} \overset{v}{+} L^{\text{blue}}_{\sim r,\sim r} = L_{\sim r,\sim r}$$

(here, we have used the fact that $r = n$ and $v \neq r$, so that when we remove the r-th row and the r-th column of the matrix L, the v-th row remains the v-th row). Hence,

$$\det \left(\underbrace{L_{\sim r,\sim r}}_{= L^{\text{red}}_{\sim r,\sim r} \overset{v}{+} L^{\text{blue}}_{\sim r,\sim r}} \right) = \det \left(L^{\text{red}}_{\sim r,\sim r} \overset{v}{+} L^{\text{blue}}_{\sim r,\sim r} \right) = \det \left(L^{\text{red}}_{\sim r,\sim r} \right) + \det \left(L^{\text{blue}}_{\sim r,\sim r} \right)$$

(by the multilinearity of the determinant).

However, a similar equality holds for the # of spanning arborescences: namely, we have

$$\begin{aligned}&(\text{\# of spanning arborescences of } D \text{ rooted to } r)\\&= \left(\text{\# of spanning arborescences of } D^{\text{red}} \text{ rooted to } r\right)\\&\qquad + \left(\text{\# of spanning arborescences of } D^{\text{blue}} \text{ rooted to } r\right).\end{aligned}$$

Here is why: Recall that an arborescence rooted to r must satisfy $\deg^+ v = 1$ (by Statement A'6 in the Dual Arborescence Equivalence Theorem, since $v \in V \setminus \{r\}$). In other words, an arborescence rooted to r must contain exactly one arc with source v. In particular, a spanning arborescence of D rooted to r must contain either a red arc or a blue arc, but not both at the same time. In the former case, it is a spanning arborescence of D^{red}; in the latter, it is a spanning arborescence of D^{blue}. Conversely, any spanning arborescence of D^{red} or of D^{blue} rooted to r is automatically a spanning arborescence of D rooted to r. Thus,

$$\begin{aligned}&(\text{\# of spanning arborescences of } D \text{ rooted to } r)\\&= \underbrace{\left(\text{\# of spanning arborescences of } D^{\text{red}} \text{ rooted to } r\right)}_{\substack{=\det\left(L^{\text{red}}_{\sim r,\sim r}\right)\\ \text{(as we saw above)}}}\\&\qquad + \underbrace{\left(\text{\# of spanning arborescences of } D^{\text{blue}} \text{ rooted to } r\right)}_{\substack{=\det\left(L^{\text{blue}}_{\sim r,\sim r}\right)\\ \text{(as we saw above)}}}\\&= \det\left(L^{\text{red}}_{\sim r,\sim r}\right) + \det\left(L^{\text{blue}}_{\sim r,\sim r}\right) = \det\left(L_{\sim r,\sim r}\right)\end{aligned}$$

(since we proved that $\det\left(L_{\sim r,\sim r}\right) = \det\left(L^{\text{red}}_{\sim r,\sim r}\right) + \det\left(L^{\text{blue}}_{\sim r,\sim r}\right)$). That is, the MTT holds for our digraph D and its vertex r. This completes the induction step, and thus the MTT (Theorem 1.1.2) is proved. $\square$

Here is one more consequence of the MTT:

Proposition 1.1.7. Let n be a positive integer. Pick any arc a of the multidigraph K_n^{bidir}. Then, the # of Eulerian circuits of K_n^{bidir} whose first arc is a is $n^{n-2} \cdot (n-2)!^n$.

Proof. Let r be the source of the arc a. The digraph K_n^{bidir} is balanced, and each of its vertices has outdegree $n-1$. By the BEST' theorem (Theorem 1.3.5 in

Lecture 16), we have

$$\begin{aligned}
&\left(\text{\# of Eulerian circuits of } K_n^{\text{bidir}} \text{ whose first arc is } a\right) \\
&= \underbrace{\left(\text{\# of spanning arborescences of } K_n^{\text{bidir}} \text{ rooted to } r\right)}_{\substack{=n^{n-2} \\ \text{(as we saw in Lecture 18)}}} \cdot \prod_{u=1}^{n} \left(\underbrace{\deg^+ u}_{=n-1} - 1\right)! \\
&= n^{n-2} \cdot \prod_{u=1}^{n} (n-2)! = n^{n-2} \cdot (n-2)!^n,
\end{aligned}$$

qed. □

In comparison, there is no good formula known for the # of Eulerian circuits of the undirected graph K_n. For n even, this # is 0 of course (since K_n has vertices of odd degree in this case). For n odd, the # grows very fast, but little else is known about it (see `https://oeis.org/A135388` for some known values).

One more remark: In Lecture 18, we have counted the trees with n vertices (i.e., simple graphs with vertex set $\{1, 2, \ldots, n\}$ that are trees). It sounds equally natural to count the "unlabelled trees with n vertices", i.e., the equivalence classes of such trees up to isomorphism. Unfortunately, this is another "messy number": the best formula known is recursive. There is also an asymptotic formula ("Otter's formula", [Otter48]): the number of equivalence classes of n-vertex trees (up to isomorphism) is

$$\approx \beta \frac{\alpha^n}{n^{5/2}} \qquad \text{with } \alpha \approx 2.955 \text{ and } \beta \approx 0.5349.$$

1.2. The undirected MTT

The Matrix-Tree Theorem becomes simpler if we apply it to a digraph of the form G^{bidir}:

Theorem 1.2.1 (undirected Matrix-Tree Theorem)**.** Let $G = (V, E, \varphi)$ be a multigraph. Assume that $V = \{1, 2, \ldots, n\}$ for some positive integer n.

Let L be the Laplacian of the digraph G^{bidir}. Explicitly, this is the $n \times n$-matrix $L \in \mathbb{Z}^{n \times n}$ whose entries are given by

$$L_{i,j} = (\deg i) \cdot [i = j] - a_{i,j},$$

where $a_{i,j}$ is the # of edges of G that have endpoints i and j (with loops counting twice). Then:

(a) For any vertex r of G, we have

$$(\text{\# of spanning trees of } G) = \det\left(L_{\sim r, \sim r}\right).$$

(b) Let t be an indeterminate. Expand the determinant $\det(tI_n + L)$ (here, I_n denotes the $n \times n$ identity matrix) as a polynomial in t:

$$\det(tI_n + L) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t^1 + c_0 t^0,$$

where $c_0, c_1, \ldots, c_n$ are numbers. (Note that this is the characteristic polynomial of L up to substituting $-t$ for t and multiplying by a power of -1. Some of its coefficients are $c_n = 1$ and $c_{n-1} = \operatorname{Tr} L$ and $c_0 = \det L$.) Then,

$$(\text{\# of spanning trees of } G) = \frac{1}{n} c_1.$$

(c) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of L, listed in such a way that $\lambda_n = 0$ (we know that 0 is an eigenvalue of L, since L is singular). Then,

$$(\text{\# of spanning trees of } G) = \frac{1}{n} \cdot \lambda_1 \lambda_2 \cdots \lambda_{n-1}.$$

Proof. **(a)** Let r be a vertex of G. Then, Proposition 1.1.1 **(b)** in Lecture 18 shows that there is a bijection

$$\left\{\text{spanning arborescences of } G^{\text{bidir}} \text{ rooted to } r\right\} \to \{\text{spanning trees of } G\}.$$

Hence, by the bijection principle, we have

$$\begin{aligned}
&(\text{\# of spanning trees of } G) \\
&= \left(\text{\# of spanning arborescences of } G^{\text{bidir}} \text{ rooted to } r\right) \\
&= \det(L_{\sim r, \sim r}) \qquad (\text{by the Matrix-Tree Theorem (Theorem 1.1.2)}).
\end{aligned}$$

This proves Theorem 1.2.1 **(a)**.

(b) We claim that

$$c_1 = \sum_{r=1}^{n} \det(L_{\sim r, \sim r}). \tag{2}$$

Note that this is a purely linear-algebraic result, and has nothing to do with the fact that L is the Laplacian of a digraph; it holds just as well if L is replaced by any square matrix.

Once (2) is proved, Theorem 1.2.1 **(b)** will easily follow, because (2) entails

$$\begin{aligned}
\frac{1}{n} c_1 &= \frac{1}{n} \sum_{r=1}^{n} \underbrace{\det(L_{\sim r, \sim r})}_{\substack{=(\text{\# of spanning trees of } G) \\ (\text{by Theorem 1.2.1 } \textbf{(a)})}} = \frac{1}{n} \underbrace{\sum_{r=1}^{n} (\text{\# of spanning trees of } G)}_{= n \cdot (\text{\# of spanning trees of } G)} \\
&= \frac{1}{n} \cdot n\,(\text{\# of spanning trees of } G) = (\text{\# of spanning trees of } G).
\end{aligned}$$

Thus, it remains to prove (2).

A rigorous proof of (2) can be found in [21s, Proposition 6.4.29] or in `https://math.stackexchange.com/a/3989575/` (both of these references actually describe all coefficients $c_0, c_1, \ldots, c_n$ of the polynomial $\det(tI_n + L)$, not just the t^1-coefficient c_1). We shall merely outline the proof of (2) on a convenient example. We want to compute c_1. In other words, we want to compute the coefficient of t^1 in the polynomial $\det(tI_n + L)$ (since c_1 is defined to be this very coefficient). Let us say that $n = 4$, so that L has the form

$$L = \begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{pmatrix}.$$

Thus,

$$\det(tI_n + L) = \det \begin{pmatrix} t + a & b & c & d \\ a' & t + b' & c' & d' \\ a'' & b'' & t + c'' & d'' \\ a''' & b''' & c''' & t + d''' \end{pmatrix}.$$

Imagine expanding the right hand side (using the Leibniz formula) and expanding the resulting products further. For instance, the product

$$(t + a)\left(t + b'\right) d'' c'''$$

becomes $ttd''c''' + tb'd''c''' + atd''c''' + ab'd''c'''$. In the huge sum that results, we are interested in those addends that contain exactly one t, because it is precisely these addends that contribute to the coefficient of t^1 in the polynomial $\det(tI_n + L)$. Where do these addends come from? To pick up exactly one t from a product like $(t + a)(t + b') d''c'''$, we need to have at least one diagonal entry in our product (for example, we cannot pick up any t from the product $cd'b''a'''$), and we need to pick out the t from this diagonal entry (rather than, e.g., the a or b' or c'' or d'''). If we pick the r-th diagonal entry, then the rest of the product is part of the expansion of $\det(L_{\sim r, \sim r})$ (since we must not pick any further ts and thus can pretend that they are not there in the first place). Thus, the total t^1-coefficient in $\det(tI_n + L)$ will be $\sum\limits_{r=1}^{n} \det(L_{\sim r, \sim r})$. This proves (2), and thus the proof of Theorem 1.2.1 **(b)** is complete.

(c) Consider the polynomial $\det(tI_n + L)$ introduced in part **(b)**, and in particular its t^1-coefficient c_1.

It is known that the characteristic polynomial $\det(tI_n - L)$ of L is a monic polynomial of degree n, and that its roots are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of L. Hence, it can be factored as follows:

$$\det(tI_n - L) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n).$$

Substituting $-t$ for t on both sides of this equality, we obtain

$$\det(-tI_n - L) = (-t-\lambda_1)(-t-\lambda_2)\cdots(-t-\lambda_n).$$

Multiplying both sides of this equality by $(-1)^n$, we find

$$\begin{aligned}\det(tI_n + L) &= (t+\lambda_1)(t+\lambda_2)\cdots(t+\lambda_n)\\ &= (t+\lambda_1)(t+\lambda_2)\cdots(t+\lambda_{n-1})\,t \qquad (\text{since } \lambda_n = 0).\end{aligned}$$

Hence, the t^1-coefficient of the polynomial $\det(tI_n + L)$ is $\lambda_1\lambda_2\cdots\lambda_{n-1}$ (since this is clearly the t^1-coefficient on the right hand side). Since we defined c_1 to be the t^1-coefficient of the polynomial $\det(tI_n + L)$, we thus conclude that $c_1 = \lambda_1\lambda_2\cdots\lambda_{n-1}$. However, Theorem 1.2.1 **(b)** yields

$$(\text{\# of spanning trees of } G) = \frac{1}{n}\underbrace{c_1}_{=\lambda_1\lambda_2\cdots\lambda_{n-1}} = \frac{1}{n}\cdot\lambda_1\lambda_2\cdots\lambda_{n-1}.$$

This proves Theorem 1.2.1 **(c)**. □

Laplacians of digraphs often have computable eigenvalues, so Theorem 1.2.1 **(c)** is actually pretty useful. A striking example of a # of spanning trees (specifically, of the n-hypercube graph Q_n, which we already met in Lecture 6) that can be counted using eigenvalues will appear on homework set #7 Exercise 5.

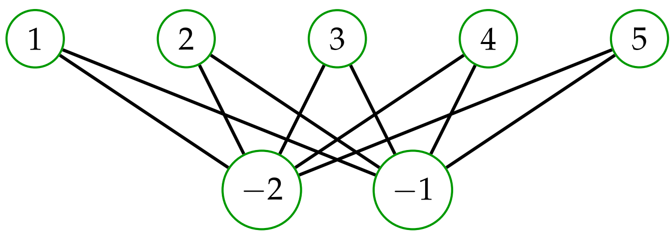
Here, however, let us give a simpler example, in which Theorem 1.2.1 **(a)** suffices:

Exercise 1. Let n and m be two positive integers. Let $K_{n,m}$ be the simple graph with $n+m$ vertices

$$1, 2, \ldots, n \qquad \text{and} \qquad -1, -2, \ldots, -m,$$

where two vertices i and j are adjacent if and only if they have opposite signs (i.e., each positive vertex is adjacent to each negative vertex, but no two vertices of the same sign are adjacent).

[For example, here is how $K_{5,2}$ looks like:



.]

How many spanning trees does $K_{n,m}$ have?

Solution. If we rename the negative vertices $-1, -2, \ldots, -m$ as $n+1, n+2, \ldots, n+m$, then the Laplacian L of the digraph $K_{n,m}^{\text{bidir}}$ can be written in block-matrix notation as follows:

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

- A is a diagonal $n \times n$-matrix whose all diagonal entries are equal to m (since there are no edges between positive vertices, and since each positive vertex has degree m);
- B is an $n \times m$-matrix whose all entries equal -1;
- C is an $m \times n$-matrix whose all entries equal -1;
- D is a diagonal $m \times m$-matrix whose all diagonal entries are equal to n.

For instance, if $n = 3$ and $m = 2$, then

$$L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{pmatrix}.$$

Theorem 1.2.1 **(a)** yields

$$(\text{\# of spanning trees of } K_{n,m}) = \det\left(L_{\sim r, \sim r}\right) \qquad \text{for any vertex } r \text{ of } K_{n,m};$$

thus, we need to compute $\det\left(L_{\sim r, \sim r}\right)$ for some vertex r. We let $r = 1$. Then, the submatrix $L_{\sim r, \sim r} = L_{\sim 1, \sim 1}$ of L again can be written in block-matrix notation as follows:

$$L_{\sim r, \sim r} = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{pmatrix}, \qquad (3)$$

where

- $\widetilde{A}$ is a diagonal $(n-1) \times (n-1)$-matrix, whose all diagonal entries are equal to m;
- $\widetilde{B}$ is an $(n-1) \times m$-matrix whose all entries equal -1;
- $\widetilde{C}$ is an $m \times (n-1)$-matrix whose all entries equal -1;
- D is a diagonal $m \times m$-matrix whose all diagonal entries are equal to n.

Fortunately, determinants of block matrices are often not hard to compute, at least when some of the blocks are invertible. For example, the Schur complement provides a neat formula. Our life here is even easier, since $\widetilde{A}$ and D are multiples of identity matrices: namely, $\widetilde{A} = mI_{n-1}$ and $D = nI_m$. We perform a "blockwise row transformation" on the block matrix $L_{\sim r, \sim r} = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{pmatrix}$, specifically subtracting the $\widetilde{C}\widetilde{A}^{-1}$-multiple of the first "block row" $\begin{pmatrix} \widetilde{A} & \widetilde{B} \end{pmatrix}$ from the second "block row" $\begin{pmatrix} \widetilde{C} & D \end{pmatrix}$ (yes, this is legitimate – it's the same as left-multiplying by the block matrix $\begin{pmatrix} I_{n-1} & 0 \\ -\widetilde{C}\widetilde{A}^{-1} & I_m \end{pmatrix}$, which has determinant 1 because it is lower-triangular). As a result, we obtain

$$\det \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{pmatrix} = \det \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} - \widetilde{C}\widetilde{A}^{-1}\widetilde{A} & D - \widetilde{C}\widetilde{A}^{-1}\widetilde{B} \end{pmatrix}$$
$$= \det \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ 0 & D - \widetilde{C}\widetilde{A}^{-1}\widetilde{B} \end{pmatrix}.$$

The matrix on the right is "block-upper triangular", so its determinant factors as follows:[5]

$$\det \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ 0 & D - \widetilde{C}\widetilde{A}^{-1}\widetilde{B} \end{pmatrix} = \det \widetilde{A} \cdot \det \left(D - \widetilde{C}\widetilde{A}^{-1}\widetilde{B} \right).$$

Of course, $\det \widetilde{A} = m^{n-1}$, since $\widetilde{A}$ is a diagonal matrix with $m, m, \ldots, m$ on the diagonal. Computing $\det \left(D - \widetilde{C}\widetilde{A}^{-1}\widetilde{B} \right)$ is a bit more complicated, but still doable: The matrix $\widetilde{A}^{-1}$ is a diagonal matrix with $m^{-1}, m^{-1}, \ldots, m^{-1}$ on the diagonal; thus, its role in the product $\widetilde{C}\widetilde{A}^{-1}\widetilde{B}$ is merely to multiply everything by m^{-1}. Hence, $\widetilde{C}\widetilde{A}^{-1}\widetilde{B} = m^{-1}\widetilde{C}\widetilde{B}$. Since all entries of $\widetilde{C}$ and $\widetilde{B}$ are -1's, we see that all entries of $\widetilde{C}\widetilde{B}$ are $(n-1)$'s. Putting all of this together, we see that $D - \widetilde{C}\widetilde{A}^{-1}\widetilde{B}$ is the $m \times m$-matrix whose all diagonal entries are equal to $n - m^{-1}(n-1)$ and whose all off-diagonal entries are equal to $-m^{-1}(n-1)$. We have already computed the determinant of a matrix much like this back in our proof of Cayley's Formula (Lecture 18); let us deal with the general case:

[5] We are using the fact that if a matrix is block-triangular (with all diagonal blocks being square matrices), then its determinant is the product of the determinants of its diagonal blocks. See, e.g., `https://math.stackexchange.com/a/1221066/` or [Grinbe20, Exercise 6.29] for a proof of this fact.

Proposition 1.2.2. Let $n \in \mathbb{N}$. Let x and a be two numbers. Then,

$$\det \underbrace{\begin{pmatrix} x & a & a & \cdots & a & a \\ a & x & a & \cdots & a & a \\ a & a & x & \cdots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & a & \cdots & x & a \\ a & a & a & \cdots & a & x \end{pmatrix}}_{\substack{\text{the } n\times n\text{-matrix} \\ \text{whose diagonal entries are } x \\ \text{and whose off-diagonal entries are } a}} = (x + (n-1)\,a)\,(x-a)^{n-1}.$$

Proposition 1.2.2 can be proved using similar reasoning as the determinant in Lecture 18; we will say more about it later. For now, let us apply it to m, $n - m^{-1}(n-1)$ and $-m^{-1}(n-1)$ instead of n, x and a, to obtain

$$\begin{aligned} \det\left(D - \widetilde{C}\widetilde{A}^{-1}\widetilde{B}\right) &= \underbrace{\left(\left(n - m^{-1}(n-1)\right) + (m-1)\left(-m^{-1}(n-1)\right)\right)}_{=1} \\ &\qquad \cdot \left(\underbrace{\left(n - m^{-1}(n-1)\right) - \left(-m^{-1}(n-1)\right)}_{=n}\right)^{m-1} \\ &= n^{m-1}. \end{aligned}$$

Now, it is time to combine everything we know. Theorem 1.2.1 **(a)** yields

$$\begin{aligned} (\text{\# of spanning trees of } K_{n,m}) &= \det\left(L_{\sim r,\sim r}\right) \\ &= \det \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{pmatrix} \qquad \text{(by (3))} \\ &= \det \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ 0 & D - \widetilde{C}\widetilde{A}^{-1}\widetilde{B} \end{pmatrix} \\ &= \underbrace{\det \widetilde{A}}_{=m^{n-1}} \cdot \underbrace{\det\left(D - \widetilde{C}\widetilde{A}^{-1}\widetilde{B}\right)}_{=n^{m-1}} \\ &= m^{n-1} \cdot n^{m-1}. \end{aligned}$$

□

Thus, we have obtained the following:

Theorem 1.2.3. Let n and m be two positive integers. Let $K_{n,m}$ be the simple graph with $n+m$ vertices

$$1,2,\ldots,n \qquad \text{and} \qquad -1,-2,\ldots,-m,$$

where two vertices i and j are adjacent if and only if they have opposite signs. Then,

$$(\text{\# of spanning trees of } K_{n,m}) = m^{n-1} \cdot n^{m-1}.$$

See [AbuSbe88] for a combinatorial proof of this theorem.

As we promised, let us make a few more remarks about Proposition 1.2.2. While this proposition can be proved by fairly straightforward row transformations (first subtracting the first row from all the other rows, then factoring an $x-a$ from all the latter rows, then subtracting a times each of the latter rows to the first row to obtain a triangular matrix), it can also be viewed as a particular case of either of the following two determinantal identities:

Proposition 1.2.4. Let $n \in \mathbb{N}$. Let $a_1, a_2, \ldots, a_n$ be n numbers, and let x be a further number. Then,

$$\det \underbrace{\begin{pmatrix} x & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_1 & x & a_2 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & x & \cdots & a_{n-1} & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & x & a_n \\ a_1 & a_2 & a_3 & \cdots & a_n & x \end{pmatrix}}_{\text{an } (n+1)\times(n+1)\text{-matrix}} = \left(x + \sum_{i=1}^{n} a_i\right) \prod_{i=1}^{n} (x - a_i).$$

Proposition 1.2.5. Let $n \in \mathbb{N}$. Let $x_1, x_2, \ldots, x_n$ be n numbers, and let a be a further number. Then,

$$\det \begin{pmatrix} x_1 & a & a & \cdots & a \\ a & x_2 & a & \cdots & a \\ a & a & x_3 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & x_n \end{pmatrix} = \prod_{i=1}^{n} (x_i - a) + a \sum_{i=1}^{n} y_i,$$

where we set $y_i := \prod_{\substack{k \in \{1,2,\ldots,n\};\\ k \neq i}} (x_k - a)$ for each $i \in \{1,2,\ldots,n\}$.

Both of these propositions make good exercises in determinant evaluation. (Proposition 1.2.4 is [Grinbe20, Exercise 6.21], while Proposition 1.2.5 is `https://math.stackexchange.com/a/2112473/` .)

See [KleSta19] for more applications of the Matrix-Tree Theorem.

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