Math 530 Spring 2022, Lecture 17: Trees

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Trees and arborescences (cont'd)

1.1. The BEST theorem (cont'd)

Last time, we stated the BEST theorem in the following two versions (which are equivalent by way of reversing all arcs):

Theorem 1.1.1 (The BEST theorem)**.** Let $D = (V, A, \psi)$ be a balanced multidigraph such that each vertex has indegree > 0. Fix an arc a of D, and let r be its target. Let $\tau(D, r)$ be the number of spanning arborescences of D rooted from r. Let $\varepsilon(D, a)$ be the number of Eulerian circuits of D whose last arc is a. Then,

$$\varepsilon(D, a) = \tau(D, r) \cdot \prod_{u \in V} \left(\deg^{-} u - 1\right)!.$$

Theorem 1.1.2 (The BEST' theorem)**.** Let $D = (V, A, \psi)$ be a balanced multidigraph such that each vertex has outdegree > 0. Fix an arc a of D, and let r be its source. Let $\tau(D, r)$ be the number of spanning arborescences of D rooted to r. Let $\varepsilon(D, a)$ be the number of Eulerian circuits of D whose first arc is a. Then,

$$\varepsilon(D, a) = \tau(D, r) \cdot \prod_{u \in V} \left(\deg^{+} u - 1\right)!.$$

We sketched a proof of the BEST' theorem. Let us now expand on this sketch. First, for the sake of convenience, we state the analogue of the Arborescence Equivalence Theorem (Theorem 1.3.5 in Lecture 15) for "arborescences rooted to r" (as opposed to "arborescences rooted from r"):

Theorem 1.1.3 (The dual arborescence equivalence theorem)**.** Let $D = (V, A, \psi)$ be a multidigraph with a to-root r. Then, the following six statements are equivalent:

- **Statement A'1:** The multidigraph D is an arborescence rooted to r.
- **Statement A'2:** We have $|A| = |V| - 1$.
- **Statement A'3:** The multigraph D^{und} is a tree.
- **Statement A'4:** For each vertex $v \in V$, the multidigraph D has a unique walk from v to r.

- **Statement A'5:** If we remove any arc from D, then the vertex r will no longer be a to-root of the resulting multidigraph.
- **Statement A'6:** We have $\deg^+ r = 0$, and each $v \in V \setminus \{r\}$ satisfies $\deg^+ v = 1$.

Proof. Upon reversing all arcs of D, this turns into the original Arborescence Equivalence Theorem (Theorem 1.3.5 in Lecture 15). $\square$

We can now prove the BEST′ theorem:

Proof of Theorem 1.1.2. Some notations first:

An **outgoing arc** from a vertex u will mean an arc whose source is u. An **incoming arc** into a vertex u will mean an arc whose target is u.

An a**-Eulerian circuit** shall mean a Eulerian circuit of D whose first arc is a.

A **sparb** shall mean a spanning arborescence of D rooted to r.

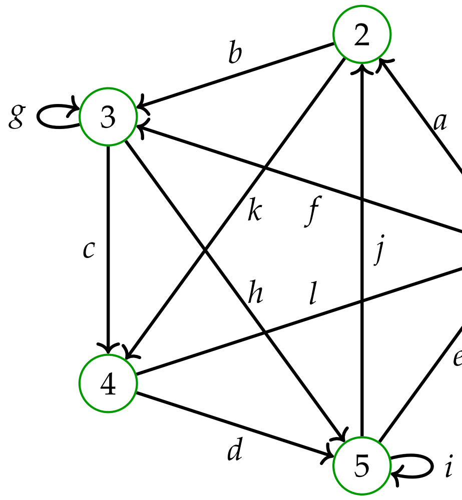
A spanning subdigraph of D always has the form $(V, B, \psi\mid_B)$ for some subset B of A. Thus, it is uniquely determined by its arc set B.

Hence, from now on, we shall identify a spanning subdigraph $(V, B, \psi\mid_B)$ of D with its arc set B. Conversely, any subset B of A will be identified with the corresponding spanning subdigraph $(V, B, \psi\mid_B)$ of D. Thus, for instance, when we say that a subset B of A "is a sparb", we shall actually mean that the corresponding spanning subdigraph $(V, B, \psi\mid_B)$ is a sparb.

For each a-Eulerian circuit $\mathbf{e}$, we define a subset $\operatorname{Exit}\mathbf{e}$ of A as follows:

Let $\mathbf{e}$ be an a-Eulerian circuit. Its first arc is a; thus, its first and last vertex is r. Being a Eulerian circuit, $\mathbf{e}$ must contain each arc of D and therefore also contain each vertex of D (since each vertex of D has outdegree > 0). For each vertex $u \in V \setminus \{r\}$, we let $e(u)$ be the **last exit** of $\mathbf{e}$ from u; this means the last arc of $\mathbf{e}$ that has source u. We let $\operatorname{Exit}\mathbf{e}$ be the set of these last exits $e(u)$ for all $u \in V \setminus \{r\}$. Thus, we have defined a subset $\operatorname{Exit}\mathbf{e}$ of A for each a-Eulerian circuit $\mathbf{e}$.

Example 1.1.4. Here is an example of this construction: Let D be the multidigraph



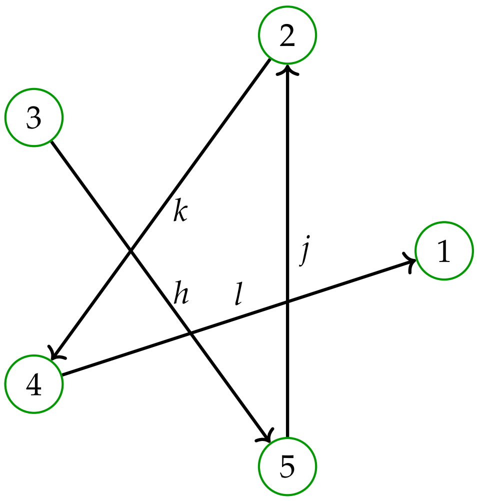
with $r = 1$, and let $\mathbf{e}$ be the a-Eulerian circuit

$$(1, a, 2, b, 3, c, 4, d, 5, e, 1, f, 3, g, 3, h, 5, i, 5, j, 2, k, 4, l, 1)$$

(we have deliberately named the arcs in such a way that they appear on a Eulerian circuit in alphabetic order). Then,

$$e(2) = k, \qquad e(3) = h, \qquad e(4) = l, \qquad e(5) = j,$$

so that $\operatorname{Exit} \mathbf{e} = \{k, h, l, j\}$. Here is $\operatorname{Exit} \mathbf{e}$ as a spanning subdigraph:



Now, we claim the following:

Claim 1: Let $\mathbf{e}$ be an a-Eulerian circuit. Then, the set $\operatorname{Exit} \mathbf{e}$ is a sparb.

Claim 2: For each sparb B (regarded as a subset of A), there are exactly $\prod_{u \in V} (\deg^+ u - 1)!$ many a-Eulerian circuits $\mathbf{e}$ such that $\operatorname{Exit} \mathbf{e} =$

B.

[*Proof of Claim 1:* The set $\operatorname{Exit} \mathbf{e}$ contains exactly one outgoing arc (namely, $e(u)$) from each vertex $u \in V \setminus \{r\}$, and no outgoing arc from r. Thus, $|\operatorname{Exit} \mathbf{e}| = |V| - 1$.

Let us number the arcs of $\mathbf{e}$ as $a_1, a_2, \ldots, a_m$, in the order in which they appear in $\mathbf{e}$. (Thus, $a_1 = a$, since the first arc of $\mathbf{e}$ is a.)

Recall that the arcs in $\operatorname{Exit} \mathbf{e}$ are the arcs $e(u)$ for all $u \in V \setminus \{r\}$ (defined as above – i.e., the arc $e(u)$ is the last exit of $\mathbf{e}$ from u). We shall refer to these arcs as the **last-exit arcs**.

For each $u \in V \setminus \{r\}$, we let $j(u)$ be the unique number $i \in \{1, 2, \ldots, m\}$ such that $e(u) = a_i$. (This i indeed exists and is unique, since each arc of D appears exactly once on $\mathbf{e}$.) Thus, $j(u)$ tells us how late in the Eulerian circuit $\mathbf{e}$ the arc $e(u)$ appears. Since $e(u)$ is the last exit of $\mathbf{e}$ from u, the Eulerian circuit $\mathbf{e}$ never visits the vertex u again after this.

Thus, if a last-exit arc $e(u)$ has target $v \neq r$, then

$$j(u) < j(v) \tag{1}$$

(because the arc $e(u)$ leads the circuit $\mathbf{e}$ into the vertex v, which the circuit then has to exit at least once; therefore, the corresponding last-exit arc $e(v)$ has to appear later in $\mathbf{e}$ than the arc $e(u)$).

We shall now show that r is a to-root of $\operatorname{Exit} \mathbf{e}$ (that is, of the spanning subdigraph $(V, \operatorname{Exit} \mathbf{e}, \psi\mid_{\operatorname{Exit} \mathbf{e}})$). To this purpose, we must show that for each vertex $v \in V$, there is a path from v to r in the digraph $(V, \operatorname{Exit} \mathbf{e}, \psi\mid_{\operatorname{Exit} \mathbf{e}})$.

Indeed, let $v \in V$ be any vertex. We must find a path from v to r in the digraph $(V, \operatorname{Exit} \mathbf{e}, \psi\mid_{\operatorname{Exit} \mathbf{e}})$. It will suffice to find a walk from v to r in this digraph (by Corollary 1.2.8 in Lecture 10). In other words, we must find a way to walk from v to r in D using last-exit arcs only.

So we start walking at v. If $v = r$, then we are already done. Otherwise, we have $v \in V \setminus \{r\}$, so that the arc $e(v)$ and the number $j(v)$ are well-defined. We thus take the arc $e(v)$. This brings us to a vertex v' (namely, the target of $e(v)$) that satisfies $j(v) < j(v')$ (by (1)). If this vertex v' is r, then we are done. If not, then $e(v')$ and $j(v')$ are well-defined, so we continue our walk by taking the arc $e(v')$. This brings us to a further vertex v'' (namely, the target of $e(v')$) that satisfies $j(v') < j(v'')$ (by (1)). If this vertex v'' is r, then we are done. Otherwise, we proceed as before. We thus construct a walk

$$(v, e(v), v', e(v'), v'', e(v''), \ldots)$$

that either goes on indefinitely or stops at the vertex r.

However, this walking process cannot go on forever (since the chain of inequalities $j(v) < j(v') < j(v'') < \cdots$ would force the numbers $j(v), j(v'), j(v''), \ldots$ to be all distinct, but there are only m distinct numbers in $\{1, 2, \ldots, m\}$). Thus, it must stop at the vertex r. So we have found a walk from v to r using last-exit arcs only. Thus, $\operatorname{Exit} \mathbf{e}$ has a walk from v to r. Hence, $\operatorname{Exit} \mathbf{e}$ has a path from v to r.

Forget that we fixed v. We thus have shown that for each vertex $v \in V$, there is a path from v to r in the digraph $(V, \operatorname{Exit} \mathbf{e}, \psi\mid_{\operatorname{Exit} \mathbf{e}})$. In other words, r is a to-root of $\operatorname{Exit} \mathbf{e}$. Hence, we conclude (using the implication A'2$\Longrightarrow$A'1 in Theorem 1.1.3) that $\operatorname{Exit} \mathbf{e}$ is an arborescence rooted to r (since $|\operatorname{Exit} \mathbf{e}| = |V| - 1$). Therefore, $\operatorname{Exit} \mathbf{e}$ is a sparb. This proves Claim 1.]

[*Proof of Claim 2:* Let B be a sparb. (As before, B is a set of arcs, and we identify it with the spanning subdigraph $(V, B, \psi\mid_B)$.)

We must prove that there are exactly $\prod_{u \in V} (\deg^+ u - 1)!$ many a-Eulerian circuits $\mathbf{e}$ such that $\operatorname{Exit} \mathbf{e} = B$.

We shall refer to the arcs in B as the B**-arcs**. Recall that B is an arborescence rooted to r (since B is a sparb). Hence, by the implication A'1$\Longrightarrow$A'6 in Theorem 1.1.3, we see that the outdegrees of its vertices satisfy

$$\deg_B^+ r = 0, \qquad \text{and} \qquad \deg_B^+ v = 1 \text{ for all } v \in V \setminus \{r\}$$

(where $\deg_B^+ v$ means the outdegree of a vertex in the digraph $(V, B, \psi\mid_B)$). In other words, there is no B-arc with source r; however, for each vertex $u \in V \setminus \{r\}$, there is exactly one B-arc with source u.

Now, we are trying to count the a-Eulerian circuits $\mathbf{e}$ such that $\operatorname{Exit} \mathbf{e} = B$.

Let us try to construct such an a-Eulerian circuit $\mathbf{e}$ as follows:

A turtle wants to walk through the digraph D using each arc of D at most once. It starts its walk by heading out from the vertex r along the arc a. From that point on, it proceeds in the usual way you would walk on a digraph: Each time it reaches a vertex, it chooses an arbitrary arc leading out of this vertex, observing the following two rules:

1. It never uses an arc that it has already used before.

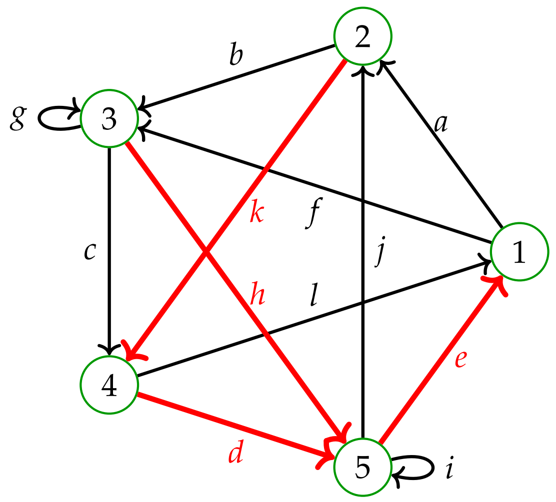
2. It never uses a B-arc unless it has to (i.e., unless this B-arc is the only outgoing arc from its current position that is still unused).

Clearly, the turtle will eventually get stuck at some vertex (with no more arcs left to continue walking along), since D has only finitely many arcs.

Let $\mathbf{w}$ be the total walk that the turtle has traced by the time it got stuck. Thus, $\mathbf{w}$ is a trail (i.e., a walk that uses no arc more than once) that starts with the vertex r and the arc a.

We will soon see that $\mathbf{w}$ is an a-Eulerian circuit satisfying $\operatorname{Exit} \mathbf{w} = B$. First, however, let us see an example:

Example 1.1.5. Let D be the multidigraph



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and let $r = 1$ and $a = a$ (we called it a on purpose). Let B be the set $\{d, e, h, k\}$, regarded as a spanning subdigraph of D. (The arcs of B are drawn bold and in red in the above picture.)

The turtle starts at $r = 1$ and walks along the arc a. This leads it to the vertex 2. It now must choose between the arcs b and k, but since it must not use the B-arc k unless it has to, it is actually forced to take the arc b next. This brings it to the vertex 3. It now has to choose between the arcs c, g and h, but again the arc h is disallowed because it is not yet time to use a B-arc. Let us say that it takes the arc g. This brings it back to the vertex 3. Next, the turtle must walk along c (since g is already used, while the B-arc still must wait until it is the only option). This brings it to the vertex 4. Its next step is to take the arc l to the vertex 1. From there, it follows the arc f to the vertex 3. Now, it can finally take the B-arc h, since all the other outgoing arcs from 3 have already been used. This brings it to the vertex 5. Now it has a choice between the arcs e, i and j, but the arc e is disallowed because it is a B-arc. Let us say it decides to use the arc j. This brings it to the vertex 2. From there, it takes the B-arc k to the vertex 4 (since it has no other options). From there, it continues along the B-arc d to the vertex 5. Now, it has to traverse the loop i, and then leave 5 along the B-arc e to come back to 1. At this point, the turtle is stuck, since it has nowhere left to go. The walk $\mathbf{w}$ we obtained is thus

$$\mathbf{w} = (1, a, 2, b, 3, g, 3, c, 4, l, 1, f, 3, h, 5, j, 2, k, 4, d, 5, i, 5, e, 1).$$

(Of course, other choices would have led to other walks.)

Returning to the general case, let us analyze the walk $\mathbf{w}$ traversed by the turtle.

- First, we claim that $\mathbf{w}$ is a closed walk (i.e., ends at r).

[*Proof:* Assume the contrary. Let u be the ending point of $\mathbf{w}$. Thus, u is the vertex at which the turtle gets stuck. Moreover, $u \neq r$ (since we just assumed that $\mathbf{w}$ is not a closed walk). Hence, the walk $\mathbf{w}$ enters the vertex u more often than it leaves it (since it ends but does not start at u). In other words, the turtle has entered the vertex u more often than it has left it. However, since D is balanced, we have $\deg^- u = \deg^+ u$. The turtle has entered the vertex u at most $\deg^- u$ times (because it cannot use an arc twice, but there are only $\deg^- u$ many arcs with target u). Thus, it has left the vertex u **less** than $\deg^- u$ times (because it has entered the vertex u more often than it has left it). Since $\deg^- u = \deg^+ u$, this means that the turtle has left the vertex u less than $\deg^+ u$ times. Thus, by the time the turtle has gotten stuck at u, there is at least one outgoing arc from u that has not been used by the turtle. Therefore, the turtle is not actually stuck at u. This is a contradiction. Thus, our assumption was wrong, so we have proved that $\mathbf{w}$ is a closed walk.]

In other words, $\mathbf{w}$ is a circuit. We shall next show that $\mathbf{w}$ is a Eulerian circuit.

To do so, we introduce one more piece of notation: A vertex u of D will be called **exhausted** if the turtle has used each outgoing arc from u (that is, if each outgoing arc from u is used in the circuit $\mathbf{w}$).

Since $\mathbf{w}$ is a circuit, the ending point of $\mathbf{w}$ is its starting point, i.e., the vertex r. Thus, the turtle must have gotten stuck at r. Hence, the vertex r is exhausted.

- We shall now show that **all** vertices of D are exhausted.

  [*Proof:* Assume the contrary. Thus, there exists a vertex u of D that is not exhausted. Consider this u. But B is a sparb, thus an arborescence rooted to r. Hence, r is a to-root of B. Therefore, there exists a path $\mathbf{p} = (p_0, b_1, p_1, b_2, p_2, \ldots, b_k, p_k)$ from u to r in B. Consider this path. Thus, we have $p_0 = u$ and $p_k = r$, and all the arcs $b_1, b_2, \ldots, b_k$ belong to B.

  There exists at least one $i \in \{0, 1, \ldots, k\}$ such that the vertex p_i is exhausted (for instance, $i = k$ qualifies, since $p_k = r$ is exhausted). Consider the **smallest** such i. Then, $p_i \neq p_0$ (since p_i is exhausted, but $p_0 = u$ is not). Hence, $i \neq 0$, so that $i \geq 1$. Therefore, p_{i-1} exists. Moreover, the vertex p_{i-1} is not exhausted (since i was defined to be the **smallest** element of $\{0, 1, \ldots, k\}$ such that p_i is exhausted).

  The arc b_i has source p_{i-1} and target p_i. Thus, it is an outgoing arc from p_{i-1} and incoming arc into p_i. Furthermore, it belongs to B (since all the arcs $b_1, b_2, \ldots, b_k$ belong to B).

  The digraph D is balanced; thus, $\deg^+ (p_i) = \deg^- (p_i)$.

  The vertex p_i is exhausted. In other words, the turtle has used each outgoing arc from p_i (by the definition of "exhausted"). Since the turtle never reuses an arc, this entails that the turtle has used exactly $\deg^+ (p_i)$ many outgoing arcs from p_i (since $\deg^+ (p_i)$ is the total number of outgoing arcs from p_i in D). In other words, it has used exactly $\deg^-(p_i)$ many outgoing arcs from p_i (since $\deg^+(p_i) = \deg^-(p_i)$).

However, the turtle's trajectory is a closed walk (in fact, it is the walk $\mathbf{w}$, which is closed). Thus, it must enter the vertex p_i as often as it leaves this vertex. In other words, the number of incoming arcs into p_i used by the turtle must equal the number of outgoing arcs from p_i used by the turtle. Since we just found (in the preceding paragraph) that the latter number is $\deg^-(p_i)$, we thus conclude that the former number is $\deg^-(p_i)$ as well. In other words, the turtle must have used exactly $\deg^-(p_i)$ many incoming arcs into p_i. Since $\deg^-(p_i)$ is the total number of incoming arcs into p_i in D, we thus conclude that the turtle must have used all incoming arcs into p_i (since the turtle never reuses an arc).

Hence, in particular, the turtle must have used the arc b_i (since b_i is an incoming arc into p_i). This arc b_i is an outgoing arc from p_{i-1}. But b_i is a B-arc, and thus our turtle uses this arc only as a last resort (i.e., after using all other outgoing arcs from p_{i-1}). Hence, we conclude that the turtle must have used all outgoing arcs from p_{i-1} (since it has used b_i). In other words, p_{i-1} is exhausted. But this contradicts the fact that p_{i-1} is not exhausted! This shows that our assumption was wrong, and our proof is finished.[1]].]

[1]For the sake of diversity, let me sketch a *second proof* of the same claim (i.e., that all vertices in D are exhausted)*:*

Assume the contrary. Thus, there exists a non-exhausted vertex u of D. Consider this u. Then, $u \neq r$ (since r is exhausted but u is not). Since u is not exhausted, there is at least one outgoing arc from u that the turtle has not used. Hence, the turtle has not used the B-arc outgoing from u (since the turtle never uses a B-arc before it has to). Let f be this B-arc, and let u' be its target. Thus, the turtle has not used all incoming arcs of u' (because it has not used the arc f). As a consequence, it has not used all outgoing arcs from u' either (because the turtle has left u' as often as it has entered u', but the balancedness of D entails that $\deg^-(u') = \deg^+(u')$). In other words, the vertex u' is non-exhausted.

Thus, by starting at the non-exhausted vertex u and taking the B-arc outgoing from u, we have arrived at a further non-exhausted vertex u'. Applying the same argument to u' instead of u, we can take a further B-arc and arrive at a further non-exhausted vertex u''. Continuing like this, we obtain an infinite sequence $(u, u', u'', \ldots)$ of non-exhausted vertices such that any vertex in this sequence is reached from the previous one by traveling along a B-arc. Clearly, this sequence must have two equal vertices (since D has only finitely many vertices). For example, let's say that $u'' = u'''''$. Then, if we consider only the part of the sequence between u'' and u''''', then we obtain a closed walk

$$\left(u'', *, u''', *, u'''', *, u'''''\right),$$

where each asterisk stands for some B-arc (not the same one, of course). This is a closed walk of the digraph $(V, B, \psi\mid_B)$. Since this closed walk has length > 0, it cannot be a path; therefore, it contains a cycle (by Proposition 1.2.9 in Lecture 10). Thus, we have found a cycle of the digraph $(V, B, \psi\mid_B)$. However, the digraph $(V, B, \psi\mid_B)$ is an arborescence, and thus has no cycles (because if D is an arborescence, then any cycle of D would be a cycle of D^{und}; but the multigraph D^{und} has no cycles by the definition of an arborescence). The previous two sentences contradict each other. This shows that our assumption was wrong, Thus, we have shown that all vertices of D are exhausted. In other words, the turtle has used all arcs of D. In other words, the trail $\mathbf{w}$ contains all arcs of D. Since $\mathbf{w}$ is a trail and a closed walk, this entails that $\mathbf{w}$ is a Eulerian circuit of D. Since $\mathbf{w}$ starts with r and a, this shows further that $\mathbf{w}$ is an a-Eulerian circuit. Since the turtle only used B-arcs as a last resort (and it used each B-arc eventually, because $\mathbf{w}$ is Eulerian), we have $\operatorname{Exit}\mathbf{w} = B$.

Thus, the turtle's walk has produced an a-Eulerian circuit $\mathbf{e}$ satisfying $\operatorname{Exit}\mathbf{e} = B$ (namely, the walk $\mathbf{w}$). However, this circuit depends on some decisions the turtle made during its walk. Namely, every time the turtle was at some vertex $u \in V$, it had to decide which arc to take next; this arc had to be an unused arc with source u, subject to the conditions that

1. if $u \neq r$, then the B-arc[2] has to be used last;
2. if $u = r$, then the arc a has to be used first.

Let us count how many options the turtle has had in total. To make the argument clearer, we modify the procedure somewhat: Instead of deciding ad-hoc which arc to take, the turtle should now make all these decisions before embarking on its journey. To do so, it chooses, for each vertex $u \in V$, a total order on the set of all arcs with source u, such that

1. if $u \neq r$, then the B-arc comes last in this order, and
2. if $u = r$, then the arc a comes first in this order.

Note that this total order can be chosen in $\left(\deg^+ u - 1\right)!$ many ways (since there are $\deg^+ u$ arcs with source u, and we can freely choose their order except that one of them has a fixed position). Thus, in total, there are $\prod\limits_{u \in V} \left(\deg^+ u - 1\right)!$ many options for how the turtle can choose all these orders. Once these orders have been chosen, the turtle then uses them to decide which arcs to walk along: Namely, the first time it visits the vertex u, it leaves it along the first arc (according to its chosen order); the second time, it uses the second arc; the third time, the third arc; and so on.

So the turtle has $\prod\limits_{u \in V} \left(\deg^+ u - 1\right)!$ many options, and each of these options leads to a different a-Eulerian circuit $\mathbf{e}$ (because the total orders chosen by the turtle are reflected in $\mathbf{e}$: they are precisely the orders in which the respective arcs appear in $\mathbf{e}$). Moreover, each a-Eulerian circuit $\mathbf{e}$ satisfying $\operatorname{Exit}\mathbf{e} = B$ comes from one of these options[3].

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and our proof is finished.

[2] We say "**the** B-arc", because there is exactly one B-arc with source u.

[3] *Proof.* Let $\mathbf{e}$ be an a-Eulerian circuit satisfying $\operatorname{Exit}\mathbf{e} = B$. Then, by choosing the appropriate total orders ahead of its journey, the turtle will trace this exact circuit $\mathbf{e}$. (Of course, the "appropriate total orders" are the ones dictated by $\mathbf{e}$: That is, for each vertex $u \in V$, the

Therefore, the total number of a-Eulerian circuits $\mathbf{e}$ satisfying $\operatorname{Exit}\mathbf{e} = B$ is the total number of options, which is $\prod\limits_{u \in V} \left(\deg^+ u - 1\right)!$ as we know. This proves Claim 2.]

With Claims 1 and 2 proved, we are almost done. The map

$$\begin{aligned} \{a\text{-Eulerian circuits of } D\} &\to \{\text{sparbs}\}, \\ \mathbf{e} &\mapsto \operatorname{Exit}\mathbf{e} \end{aligned}$$

is well-defined (by Claim 1). Furthermore, Claim 2 shows that this map is a $\prod\limits_{u \in V} \left(\deg^+ u - 1\right)!$-to-1 correspondence[4] (i.e., each sparb B has exactly $\prod\limits_{u \in V} \left(\deg^+ u - 1\right)!$ many preimages under this map). Thus, by the multijection principle[5], we conclude that[6]

$$(\#\text{ of } a\text{-Eulerian circuits of } D) = \left(\prod_{u \in V} \left(\deg^+ u - 1\right)!\right) \cdot (\#\text{ of sparbs}).$$

Since $\varepsilon(D, a) = (\#\text{ of } a\text{-Eulerian circuits of } D)$ and $\tau(D, r) = (\#\text{ of sparbs})$, we can rewrite this as follows:

$$\varepsilon(D, a) = \left(\prod_{u \in V} \left(\deg^+ u - 1\right)!\right) \cdot \tau(D, r) = \tau(D, r) \cdot \prod_{u \in V} \left(\deg^+ u - 1\right)!.$$

This proves Theorem 1.1.2. □

Proof of Theorem 1.1.1. As we already mentioned, Theorem 1.1.1 follows from Theorem 1.1.2 by reversing each arc (i.e., by applying Theorem 1.1.2 to the digraph D^{rev} instead of D). □

Before we actually use the BEST (or BEST′) theorem to count the Eulerian circuits on any digraph, let us mention a neat corollary for the number of spanning arborescences:

turtle must pick the same total order on the set of all arcs with source u in which they appear on $\mathbf{e}$. This choice is legitimate, because the arc a is the first arc of $\mathbf{e}$ (so it will certainly come first in its order), and because each B-arc appears in $\mathbf{e}$ after all other arcs from the same source have appeared (so it will come last in its total order).)

[4] An m**-to-**1 **correspondence** (where m is a nonnegative integer) means a map $f : X \to Y$ between two sets such that each element of Y has exactly m preimages under f.

[5] The **multijection principle** is a basic counting principle that says the following: Let X and Y be two finite sets, and let $m \in \mathbb{N}$. Let $f : X \to Y$ be an m-to-1 correspondence (i.e., a map such that each element of Y has exactly m preimages under f). Then, $|X| = m \cdot |Y|$.

For example, n (intact) sheep have $4n$ legs in total, since the map that sends each leg to its sheep is a 4-to-1 correspondence.

[6] The symbol "#" means "number".

Corollary 1.1.6. Let $D = (V, A, \psi)$ be a balanced multidigraph. For each vertex $r \in V$, let $\tau(D, r)$ be the number of spanning arborescences of D rooted to r. Then, $\tau(D, r)$ does not depend on r.

Proof of Corollary 1.1.6. WLOG assume that $|V| > 1$ (else, the claim is obvious). If there is a vertex $v \in V$ with $\deg^+ v = 0$, then this vertex v satisfies $\deg^- v = 0$ as well (since the balancedness of D entails $\deg^- v = \deg^+ v = 0$), and therefore D has no spanning arborescences at all (since any spanning arborescence would have an arc with source or target v). Thus, we WLOG assume that $\deg^+ v > 0$ for all $v \in V$. In other words, each vertex has outdegree > 0.

Let r and s be two vertices of D. We must prove that $\tau(D, r) = \tau(D, s)$.

Pick an arc a with source r. (This exists, since $\deg^+ r > 0$.) Pick an arc b with source s. (This exists, since $\deg^+ s > 0$.)

Applying the BEST' theorem (Theorem 1.1.2), we get

$$\varepsilon(D, a) = \tau(D, r) \cdot \prod_{u \in V} \left(\deg^+ u - 1\right)! \qquad \text{and similarly}$$
$$\varepsilon(D, b) = \tau(D, s) \cdot \prod_{u \in V} \left(\deg^+ u - 1\right)!.$$

However, $\varepsilon(D, a) = \varepsilon(D, b)$, since counting Eulerian circuits that start with a is equivalent to counting Eulerian circuits that start with b (because an Eulerian circuit can be rotated uniquely to start with any given arc). Thus, we obtain

$$\tau(D, r) \cdot \prod_{u \in V} \left(\deg^+ u - 1\right)! = \varepsilon(D, a) = \varepsilon(D, b) = \tau(D, s) \cdot \prod_{u \in V} \left(\deg^+ u - 1\right)!.$$

Cancelling the (nonzero!) number $\prod_{u \in V} \left(\deg^+ u - 1\right)!$ from this equality, we obtain $\tau(D, r) = \tau(D, s)$. This proves Corollary 1.1.6. □