Math 530 Spring 2022, Lecture 16: Trees

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Trees and arborescences (cont'd)

1.1. Arborescences vs. trees

Last time, we defined from-roots and arborescences:

Definition 1.1.1. Let D be a multidigraph. Let r be a vertex of D.

(a) We say that r is a **from-root** (or, short, **root**) of D if for each vertex v of D, the digraph D has a path from r to v.

(b) We say that D is an **arborescence rooted from** r if r is a from-root of D and the undirected multigraph D^{und} has no cycles.

Last time, we proved the arborescence equivalence theorem, which gave several equivalent criteria for a digraph with a from-root r to be an arborescence rooted from r. We furthermore claimed the following theorem, connecting arborescences with trees:

Theorem 1.1.2. Let D be a multidigraph, and let r be a vertex of D. Then, the following two statements are equivalent:

- **Statement C1:** The multidigraph D is an arborescence rooted from r.
- **Statement C2:** The undirected multigraph D^{und} is a tree, and each arc of D is "oriented away from r" (this means the following: the source of this arc lies on the unique path between r and the target of this arc on D^{und}).

Our next goal is to prove this theorem. It is an easy theorem to believe, but an annoyingly hard one to formally prove!

To prove it formally, we introduce a few notations regarding trees. First, a simple proposition:

Proposition 1.1.3. Let $T = (V, E, \varphi)$ be a tree. Let $r \in V$ be a vertex of T. Let e be an edge of T, and let u and v be its two endpoints. Then, the distances $d(r,u)$ and $d(r,v)$ differ by exactly 1 (that is, we have either $d(r,u) = d(r,v) + 1$ or $d(r,v) = d(r,u) + 1$).

Proof. We recall that since T is a tree, the distance $d(p, q)$ between two vertices p and q of T is simply the length of the path from p to q. (This path is unique, since T is a tree.)

Let $\mathbf{p}$ be the path from r to u. Then, we are in one of the following two cases:

Case 1: The edge e is an edge of $\mathbf{p}$.

Case 2: The edge e is not an edge of $\mathbf{p}$.

Consider Case 1. In this case, e must be the **last** edge of $\mathbf{p}$ (since otherwise, $\mathbf{p}$ would visit u more than once, but $\mathbf{p}$ cannot do this, since $\mathbf{p}$ is a path). Thus, if we remove this last edge e (and the vertex u) from $\mathbf{p}$, then we obtain a path from r to v. This path is exactly one edge shorter than $\mathbf{p}$. Thus, $d(r, v) = d(r, u) - 1$, so that $d(r, u) = d(r, v) + 1$. So we are done in Case 1.

Now, consider Case 2. In this case, the edge e is not an edge of $\mathbf{p}$. Thus, we can append e and v to the end of the path $\mathbf{p}$, and the result will be a backtrack-free walk $\mathbf{p}'$. However, a backtrack-free walk in a tree is always a path (since otherwise, it would contain a cycle[1], but a tree has no cycles). Thus, $\mathbf{p}'$ is a path from r to v, and it is exactly one edge longer than $\mathbf{p}$ (by its construction). Therefore, $d(r, v) = d(r, u) + 1$. So we are done in Case 2.

Now, we are done in both cases, so that Proposition 1.1.3 is proven. □

Definition 1.1.4. Let $T = (V, E, \varphi)$ be a tree. Let $r \in V$ be a vertex of T. Let e be an edge of T. By Proposition 1.1.3, the distances from the two endpoints of e to the vertex r differ by exactly 1. So one of them is smaller than the other.

(a) We define the r**-parent** of e to be the endpoint of e whose distance to r is the smallest. We denote this endpoint by e^{-r}.

(b) We define the r**-child** of e to be the endpoint of e whose distance to r is the largest. We denote this endpoint by e^{+r}.

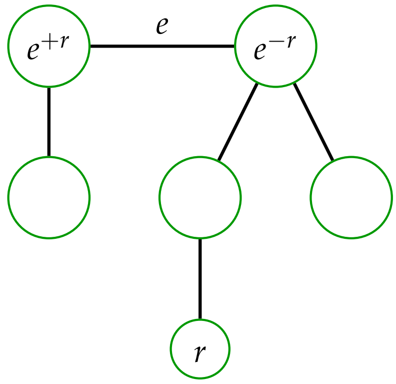
Thus, by Proposition 1.1.3, we have

$$d\left(r, e^{+r}\right) = d\left(r, e^{-r}\right) + 1.$$

Example 1.1.5. Here is a tree T, a vertex r, an edge e and its r-parent e^{-r} and

[1] by Proposition 1.1.2 in Lecture 13

its r-child e^{+r}:

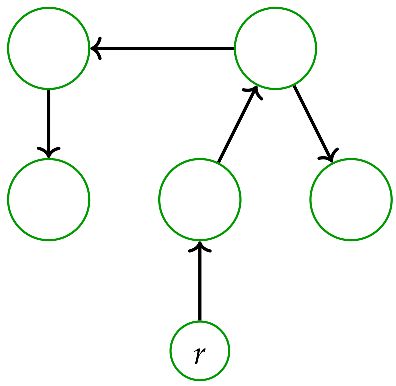


Definition 1.1.6. Let $T = (V, E, \varphi)$ be a tree. Let $r \in V$ be a vertex of T. Then, we define a multidigraph $T^{r\to}$ by

$$T^{r\to} := (V, E, \psi),$$

where $\psi : E \to V \times V$ is the map that sends each edge $e \in E$ to the pair (e^{-r}, e^{+r}). Colloquially speaking, this means that $T^{r\to}$ is the multidigraph obtained from T by turning each edge e into an arc from its r-parent e^{-r} to its r-child e^{+r}. This is what we mean when we speak of "orienting each edge of T away from r" in Theorem 1.1.2.

Example 1.1.7. If T is the tree from Example 1.1.5, then $T^{r\to}$ is the following multidigraph:



Now, Theorem 1.1.2 can be rewritten as follows:

Theorem 1.1.8. Let D be a multidigraph, and let r be a vertex of D. Then, the following two statements are equivalent:

- **Statement C1:** The multidigraph D is an arborescence rooted from r.
- **Statement C2:** The undirected multigraph D^{und} is a tree, and we have $D = \left(D^{\text{und}}\right)^{r\to}$. (This is a honest equality, not just some isomorphism.)

The proof of this theorem is best organized by splitting into two lemmas:

Lemma 1.1.9. Let $T = (V, E, \varphi)$ be a tree. Let $r \in V$ be a vertex of T. Then, the multidigraph $T^{r\to}$ is an arborescence rooted from r.

Proof. The idea is to show that if $\mathbf{p}$ is a path from r to some vertex v in the tree T, then $\mathbf{p}$ is also a path in the digraph $T^{r\to}$, because all the edges of $\mathbf{p}$ have been "oriented correctly" (i.e., their orientation matches how they are used in $\mathbf{p}$).

Here are the details: Clearly, $(T^{r\to})^{\text{und}} = T$. Hence, the graph $(T^{r\to})^{\text{und}}$ is a tree and hence has no cycles. Thus, it suffices to prove that r is a from-root of $T^{r\to}$. In other words, we must prove that

$$T^{r\to} \text{ has a path from } r \text{ to } v \tag{1}$$

for each $v \in V$.

We shall prove (1) by induction on $d(r, v)$ (where d means the distance on the tree T):

Base case: If $v \in V$ satisfies $d(r, v) = 0$, then $v = r$, and thus $T^{r\to}$ has a path from r to v (namely, the trivial path (r)). Thus, (1) is proved for $d(r, v) = 0$.

Induction step: Let $k \in \mathbb{N}$. Assume (as the induction hypothesis) that (1) holds for each $v \in V$ satisfying $d(r, v) = k$. We must now prove the same for each $v \in V$ satisfying $d(r, v) = k + 1$.

So let $v \in V$ satisfy $d(r, v) = k + 1$. Then, the path of T from r to v has length $k + 1$. Let $\mathbf{p}$ be this path, let e be its last edge, and let u be its second-to-last vertex (so that its last edge e has endpoints u and v). Then, by removing the last edge e from the path $\mathbf{p}$, we obtain a path from r to u that is one edge shorter than $\mathbf{p}$. Hence, $d(r, u) = d(r, v) - 1 < d(r, v)$. Consequently, the edge e has r-parent u and r-child v (by Definition 1.1.4). In other words, $e^{-r} = u$ and $e^{+r} = v$. Therefore, in the digraph $T^{r\to}$, the edge e is an arc from u to v (by Definition 1.1.6). Moreover, we have $d(r, u) = d(r, v) - 1 = k$ (since $d(r, v) = k + 1$); therefore, the induction hypothesis tells us that (1) holds for u instead of v. In other words, $T^{r\to}$ has a path from r to u. Attaching the arc e and the vertex v to this path, we obtain a walk of $T^{r\to}$ from r to v (since e is an arc from u to v in $T^{r\to}$). Thus, the digraph $T^{r\to}$ has a walk from r to v, therefore also a path from r to v. Hence, (1) holds for our v. This completes the induction step.

Thus, (1) is proved by induction. As we explained above, this yields Lemma 1.1.9. □

Lemma 1.1.10. Let $D = (V, A, \psi)$ be an arborescence rooted from r (for some $r \in V$). Let $a \in A$ be an arc of D. Let s be the source of a, and let t be the target of a. Then:

- **(a)** We have $d(r, s) < d(r, t)$, where d means distance on the tree D^{und}.
- **(b)** In the multidigraph $\left(D^{\text{und}}\right)^{r\to}$, the arc a has source s and target t.

Proof. **(a)** The vertex r is a from-root of D (since D is an arborescence rooted from r). Thus, D has a path from r to t. Let $\mathbf{p}$ be this path. Note that $\deg^- t \geq 1$, since t is the target of at least one arc (namely, of a).

The digraph D is an arborescence rooted from r, and thus satisfies Statement A6 in the arborescence equivalence theorem (Theorem 1.3.5 in Lecture 15). In other words, we have

$$\deg^- r = 0 \qquad \text{and} \qquad \deg^- v = 1 \text{ for each } v \in V \setminus \{r\}.$$

In particular, this entails $\deg^- v \leq 1$ for each $v \in V$. Applying this to $v = t$, we obtain $\deg^- t \leq 1$. Hence, the arc a is the **only** arc whose target is t.

We have $t \neq r$ (since $\deg^- r = 0$ but $\deg^- t \geq 1 > 0$). Thus, the path $\mathbf{p}$ from r to t has at least one arc. Its last arc is therefore an arc whose target is t. Hence, this last arc is a (since a is the **only** arc whose target is t).

If we remove this last arc from the path $\mathbf{p}$, then we obtain a path $\mathbf{p}'$ from r to s (since s is the source of a).

However, each path of D is a path of D^{und}. Thus, in particular, $\mathbf{p}$ is a path of D^{und} from r to t, while $\mathbf{p}'$ is a path of D^{und} from r to s. Since $\mathbf{p}'$ is exactly one edge shorter than $\mathbf{p}$, we thus obtain $d(r,s) = d(r,t) - 1 < d(r,t)$. This proves Lemma 1.1.10 **(a)**.

(b) The arc a of the digraph D has source s and target t. Hence, the edge a of the tree D^{und} has endpoints s and t. Since $d(r,s) < d(r,t)$ (by part **(a)**), this entails that its r-parent is s and its r-child is t (by Definition 1.1.4). Thus, in the digraph $\left(D^{\text{und}}\right)^{r\to}$, this edge a becomes an arc with source s and target t (by Definition 1.1.6). This proves Lemma 1.1.10 **(b)**. □

Proof of Theorem 1.1.8. If (V, A, ψ) is a multidigraph, then we shall refer to the map $\psi : A \to V \times V$ (which determines the source and the target of each arc) as the "psi-map" of this multidigraph.

Write the multidigraph D as $D = (V, A, \psi)$. We shall now prove the implications C1$\Longrightarrow$C2 and C2$\Longrightarrow$C1 separately:

Proof of the implication C1$\Longrightarrow$C2: Assume that Statement C1 holds. That is, D is an arborescence rooted from r. We must prove Statement C2. In other words, we must prove that the undirected multigraph D^{und} is a tree, and that $D = \left(D^{\text{und}}\right)^{r\to}$.

It is clear (by the definition of an arborescence) that D^{und} is a tree. It thus remains to prove that $D = \left(D^{\text{und}}\right)^{r\to}$.

The multidigraphs D and $\left(D^{\text{und}}\right)^{r\to}$ have the same set of vertices (namely, V) and the same set of arcs (namely, A); we therefore just need to show that their psi-maps are the same. In other words, we need to show that $\psi' = \psi$, where ψ' is the psi-map of $\left(D^{\text{und}}\right)^{r\to}$.

Let $a \in A$ be arbitrary. Let $\psi(a) = (s,t)$. Thus, the arc a of D has source s and target t. Lemma 1.1.10 **(b)** therefore shows that in the multidigraph $\left(D^{\text{und}}\right)^{r\to}$, the arc a has source s and target t as well. In other words, $\psi'(a) = (s,t)$ (since ψ' is the psi-map of this multidigraph). Hence, $\psi'(a) = (s,t) = \psi(a)$.

Forget that we fixed a. We thus have shown that $\psi'(a) = \psi(a)$ for each $a \in A$. In other words, $\psi' = \psi$. As explained above, this completes the proof of Statement C2.

Proof of the implication C2$\Longrightarrow$C1: Assume that Statement C2 holds. Thus, the undirected multigraph D^{und} is a tree, and we have $D = \left(D^{\text{und}}\right)^{r\rightarrow}$. Hence, Lemma 1.1.9 (applied to $T = D^{\text{und}}$) yields that the multidigraph $\left(D^{\text{und}}\right)^{r\rightarrow}$ is an arborescence rooted from r. In other words, D is an arborescence rooted from r (since $D = \left(D^{\text{und}}\right)^{r\rightarrow}$). This shows that Statement C1 holds.

Having now proved both implications C1$\Longrightarrow$C2 and C2$\Longrightarrow$C1, we conclude that Statements C1 and C2 are equivalent. Thus, Theorem 1.1.8 is proved. $\square$

Oof.

Let's get one more consequence out of this. First, let us show that an arborescence can have only one root:

Proposition 1.1.11. Let D be an arborescence rooted from r. Then, r is the **only** root of D.

Proof of Proposition 1.1.11. Assume the contrary. Thus, D has another root s distinct from r. Hence, D has a path from r to s (since r is a root) as well as a path from s to r (since s is a root). Combining these paths gives a circuit of length > 0. However, a circuit of length > 0 in a digraph must always contain a cycle (since Proposition 1.2.9 in Lecture 10 shows that it either is a path or contains a cycle; but it clearly cannot be a path). Hence, D has a cycle. Therefore, D^{und} also has a cycle (since any cycle of D is a cycle of D^{und}). However, D^{und} has no cycles (since D is an arborescence rooted from r). The preceding two sentences contradict each other. This shows that the assumption was wrong, and Proposition 1.1.11 is proven. $\square$

Definition 1.1.12. A multidigraph D is said to be an **arborescence** if there exists a vertex r of D such that D is an arborescence rooted from r. In this case, this r is uniquely determined as the only root of D (by Proposition 1.1.11).

Theorem 1.1.13. There are two mutually inverse maps

$$\begin{aligned} \{\text{pairs } (T,r) \text{ of a tree } T \text{ and a vertex } r \text{ of } T\} &\to \{\text{arborescences}\}, \\ (T,r) &\mapsto T^{r\rightarrow} \end{aligned}$$

and

$$\begin{aligned} \{\text{arborescences}\} &\to \{\text{pairs } (T,r) \text{ of a tree } T \text{ and a vertex } r \text{ of } T\}, \\ D &\mapsto \left(D^{\text{und}}, \sqrt{D}\right), \end{aligned}$$

where $\sqrt{D}$ denotes the root of D.

Proof. The map

$$\{\text{pairs } (T,r) \text{ of a tree } T \text{ and a vertex } r \text{ of } T\} \to \{\text{arborescences}\}, \\ (T,r) \mapsto T^{r\to}$$

is well-defined because of Lemma 1.1.9. The map

$$\{\text{arborescences}\} \to \{\text{pairs } (T,r) \text{ of a tree } T \text{ and a vertex } r \text{ of } T\}, \\ D \mapsto \left(D^{\text{und}}, \sqrt{D}\right),$$

is well-defined because if D is an arborescence, then D^{und} is a tree. In order to show that these two maps are mutually inverse, we must check the following two statements:

1. Each arborescence D satisfies $\left(D^{\text{und}}\right)^{r\to} = D$, where r is the root of D;
2. Each pair (T,r) of a tree T and a vertex r of T satisfies $\left(T^{r\to}\right)^{\text{und}} = T$ and $\sqrt{\left(T^{r\to}\right)^{\text{und}}} = r$.

However, Statement 1 follows from Theorem 1.1.8 (specifically, from the implication C1$\Longrightarrow$C2 in Theorem 1.1.8). Statement 2 follows from Lemma 1.1.9 (more precisely, the $\left(T^{r\to}\right)^{\text{und}} = T$ part of Statement 2 is obvious, whereas the $\sqrt{\left(T^{r\to}\right)^{\text{und}}} = r$ part follows from Lemma 1.1.9). Thus, Theorem 1.1.13 is proved. $\square$

Theorem 1.1.13 formalizes the idea that an arborescence is "just a tree with a chosen vertex". For this reason, arborescences are sometimes called "oriented trees", but this name is also shared with a more general notion, which is why I avoid it.

1.2. Spanning arborescences

In analogy to spanning subgraphs of a multigraph, we can define spanning subdigraphs of a multidigraph:

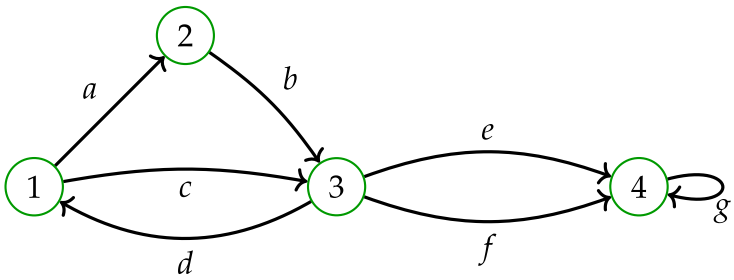
Definition 1.2.1. A **spanning subdigraph** of a multidigraph $D = (V, A, \psi)$ means a multidigraph of the form $(V, B, \psi\mid_B)$, where B is a subset of A.

In other words, it means a submultidigraph of D with the same vertex set as D.

In other words, it means a multidigraph obtained from D by removing some arcs, but leaving all vertices untouched.

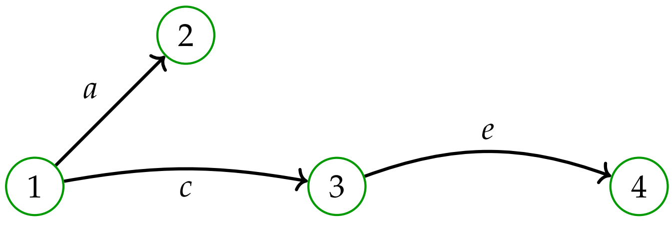
Definition 1.2.2. Let D be a multidigraph. Let r be a vertex of D. A **spanning arborescence of** D **rooted from** r means a spanning subdigraph of D that is an arborescence rooted from r.

Example 1.2.3. Let $D = (V, A, \psi)$ be the following multidigraph:

$D =$ 

.

Is there a spanning arborescence of D rooted from 1 ? Yes, for instance,

$\left(V,\ \{a,c,e\},\ \psi\mid_{\{a,c,e\}}\right) =$ 

.

By abuse of notation, we shall refer to this spanning arborescence simply as $\{a,c,e\}$ (since a spanning subdigraph of D is uniquely determined by its arc set). Another spanning arborescence of D rooted from 1 is $\{a,b,e\}$. Yet another is $\{a,b,f\}$. A non-example is $\{a,d,f\}$ (indeed, this is an arborescence rooted from 3, not from 1).

Is there a spanning arborescence of D rooted from 2 ? Yes, for example $\{b,d,f\}$.

Is there a spanning arborescence of D rooted from 4 ? No, since 4 is not a from-root of D.

This illustrates a first obstruction to the existence of spanning arborescences: Namely, a digraph D can have a spanning arborescence rooted from r only if r is a from-root. This necessary criterion is also sufficient:

Theorem 1.2.4. Let D be a multidigraph. Let r be a from-root of D. Then, D has a spanning arborescence rooted from r.

Proof. This is an analogue of the "every connected multigraph has a spanning tree" theorem that we proved in 4 ways in Lectures 14 and 15. At least the first proof easily adapts to the directed case:

Remove arcs from D one by one, but in such a way that the "rootness of r" (that is, the property that r is a root of our multidigraph) is preserved. So we can only remove an arc if r remains a root afterwards.

Clearly, this removing process will eventually come to an end, since D has only finitely many arcs. Let D' be the multidigraph obtained at the end of this process. Then, r is still a root of D', but we cannot remove any more arcs from D' without breaking the rootness of r. That is, if we remove any arc from D', then the vertex r will no longer be a from-root of the resulting multidigraph. This means that D' satisfies Statement A5 from the arborescence equivalence theorem (Theorem 1.3.5 in Lecture 15). Thus, D' satisfies Statement A1 as well (since all six statements A1, A2, ..., A6 are equivalent). In other words, D' is an arborescence rooted from r. Since D' is a spanning subdigraph of D, we thus conclude that D has a spanning arborescence rooted from r (namely, D'). This proves Theorem 1.2.4. □

Question 1.2.5. Can we adapt the other three proofs too?

1.3. The BEST theorem

Recall that a multidigraph $D = (V, A, \varphi)$ is **balanced** if and only if each vertex v satisfies $\deg^- v = \deg^+ v$. This is necessary for the existence of a Eulerian circuit. If D is weakly connected, this is also sufficient (by Exercise 3 **(a)** on homework set #4).

Surprisingly, there is a formula for the number of these Eulerian circuits:

Theorem 1.3.1 (The BEST theorem)**.** Let $D = (V, A, \psi)$ be a balanced multidigraph such that each vertex has indegree > 0. Fix an arc a of D, and let r be its target. Let $\tau(D, r)$ be the number of spanning arborescences of D rooted from r. Let $\varepsilon(D, a)$ be the number of Eulerian circuits of D whose last arc is a. Then,

$$\varepsilon(D, a) = \tau(D, r) \cdot \prod_{u \in V} \left(\deg^- u - 1\right)!.$$

The "BEST" in the name of this theorem is an abbreviation for de Bruijn, van Aardenne–Ehrenfest, Smith and Tutte.[2]

To prove this theorem, we shall restate it in terms of "arborescences to" (as opposed to "arborescences from"). Mathematically speaking, this restatement isn't really necessary (the argument is the same in both cases up to reversing the directions of all arcs), but it helps make the proof more intuitive, since it lets us build our Eulerian circuits by moving forwards rather than backwards.

Here is the formal definition of "arborescences to":

[2] We note that the number of Eulerian circuits of D whose last arc is a is precisely the number of all Eulerian circuits of D counted up to rotation. Indeed, each Eulerian circuit of D contains the arc a exactly once, and thus can be rotated in a unique way to end with a.

Definition 1.3.2. Let D be a multidigraph. Let r be a vertex of D.

(a) We say that r is a **to-root** of D if for each vertex v of D, the digraph D has a path from v to r.

(b) We say that D is an **arborescence rooted to** r if r is a to-root of D and the undirected multigraph D^{und} has no cycles.

Clearly, Definition 1.1.1 and Definition 1.3.2 differ only in the direction of the arcs. In other words, if we reverse each arc of our digraph (turning its source into its target and vice versa), then a from-root becomes a to-root, and an arborescence rooted from r becomes an arborescence rooted to r, and vice versa. Thus, every property that we have proved for arborescences rooted from r can be translated into the language of arborescences rooted to r by reversing all arcs.

If you want to see this stated more rigorously, here is a formal definition of "reversing each arc":

Definition 1.3.3. Let $D = (V, A, \psi)$ be a multidigraph. Then, D^{rev} shall denote the multidigraph $(V, A, \tau \circ \psi)$, where $\tau : V \times V \to V \times V$ is the map that sends each pair (s, t) to (t, s). Thus, if an arc a of D has source s and target t, then it is also an arc of D^{rev}, but in this digraph D^{rev} it has source t and target s.

The multidigraph D^{rev} is called the **reversal** of the multidigraph D; we say that it is obtained from D by "reversing each arc".

This notion of "reversing each arc" allows us to reverse walks in digraphs: If $\mathbf{w}$ is a walk from a vertex s to t in some multidigraph D, then its reversal $\operatorname{rev} \mathbf{w}$ (obtained by reading $\mathbf{w}$ backwards) is a walk from t to s in the multidigraph D^{rev}. The same holds if we replace the word "walk" by "path". Thus, we easily obtain the following:

Proposition 1.3.4. Let D be a multidigraph. Let r be a vertex of D. Then:

(a) The vertex r is a to-root of D if and only if r is a from-root of D^{rev}.

(b) The digraph D is an arborescence rooted to r if and only if D^{rev} is an arborescence rooted from r.

Proof. Completely straightforward unpacking of the definitions. $\square$

Note that when we reverse each arc in a digraph D, the outdegrees of its vertices become their indegrees and vice versa. Hence, a balanced digraph D remains balanced when this happens. In particular, the BEST theorem (Theorem 1.3.1) thus gets translated as follows:

Theorem 1.3.5 (The BEST′ theorem)**.** Let $D = (V, A, \psi)$ be a balanced multidigraph such that each vertex has outdegree > 0. Fix an arc a of D, and let r be its source. Let $\tau(D, r)$ be the number of spanning arborescences of D rooted to r. Let $\varepsilon(D, a)$ be the number of Eulerian circuits of D whose first arc is a. Then,

$$\varepsilon(D, a) = \tau(D, r) \cdot \prod_{u \in V} \left(\deg^+ u - 1\right)!.$$

Proof idea. Here is the main idea of the proof; we will see the details next time:

An a**-Eulerian circuit** shall mean a Eulerian circuit of D whose first arc is a.

Let $\mathbf{e}$ be an a-Eulerian circuit. Its first arc is a; therefore, its first and last vertex is r.

Being an Eulerian circuit, $\mathbf{e}$ must contain each arc of D and therefore contain each vertex of D (since each vertex has outdegree > 0). For each vertex $u \neq r$, we let $e(u)$ be the **last exit** of $\mathbf{e}$ from u, that is, the last arc of $\mathbf{e}$ that has source u. Let $\operatorname{Exit} \mathbf{e}$ be the set of these last exits $e(u)$ for all vertices $u \neq r$. Then, we claim:

> *Claim 1:* This set $\operatorname{Exit} \mathbf{e}$ (or, more precisely, the spanning subdigraph $(V,\ \operatorname{Exit} \mathbf{e},\ \psi\mid_{\operatorname{Exit} \mathbf{e}})$) is a spanning arborescence of D rooted to r.

Let's assume for the moment that Claim 1 is proven. Thus, given any a-Eulerian circuit $\mathbf{e}$, we have constructed a spanning arborescence of D rooted to r.

How many a-Eulerian circuits $\mathbf{e}$ lead to a given arborescence in this way? The answer is rather nice:

> *Claim 2:* For each spanning arborescence $(V, B, \psi\mid_B)$ of D rooted to r, there are exactly $\prod\limits_{u \in V} \left(\deg^+ u - 1\right)!$ many a-Eulerian circuits $\mathbf{e}$ such that $\operatorname{Exit} \mathbf{e} = B$.

Let us again assume that this is proven. Combining Claim 1 with Claim 2, we obtain a $\prod\limits_{u \in V} \left(\deg^+ u - 1\right)!$-to-1 correspondence between the a-Eulerian circuits and the spanning arborescences of D rooted to r. Thus, the number of the former is $\prod\limits_{u \in V} \left(\deg^+ u - 1\right)!$ times the number of the latter. But this is precisely the claim of Theorem 1.3.5. Hence, in order to prove Theorem 1.3.5, it remains to prove Claim 1 and Claim 2. We will do this in the next lecture. $\square$