# Math 530 Spring 2022, Lecture 12: more tournaments and Hamiltonian paths

**website:** `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

Today's material appears in more detail in:

- Lecture 7 from my Spring 2017 course ( `https://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec7.pdf` )
- Lecture 8 from my Spring 2017 course ( `https://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec8.pdf` ).

# 1. Digraphs and multidigraphs (cont'd)

## 1.1. Tournaments (cont'd)

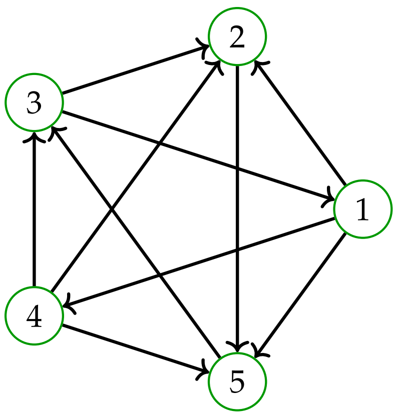
Recall some definitions from Lecture 11:

**Definition 1.1.1.** A digraph $D$ is said to be **loopless** if it has no loops.

**Definition 1.1.2.** A **tournament** is defined to be a loopless simple digraph $D$ that satisfies the

- **Tournament axiom:** For any two distinct vertices $u$ and $v$ of $D$, **exactly** one of $(u, v)$ and $(v, u)$ is an arc of $D$.

**Example 1.1.3.** Here is a tournament with 5 vertices:




.

A tournament can also be viewed as a complete graph, whose each edge has been given a direction.

Recall also the following definitions from Lecture 11:

**Definition 1.1.4.** Let $D = (V, A)$ be a simple digraph. Then:

**(a)** We define the simple digraph $D^{\text{rev}}$ to be $(V, A^{\text{rev}})$, where

$$A^{\text{rev}} = \{(v,u) \mid (u,v) \in A\}.$$

**(b)** We define the simple digraph $\overline{D}$ to be $\left(V, \overline{A}\right)$, where

$$\overline{A} = (V \times V) \setminus A.$$

The definition of a tournament can now be restated as follows:

**Proposition 1.1.5.** Let $D = (V, A)$ be a loopless simple digraph. Then, $D$ is a tournament if and only if the non-loop arcs of $\overline{D}$ are precisely the arcs of $D^{\text{rev}}$.

In Lecture 11, we stated the following two theorems:[1]

**Theorem 1.1.6** (Easy Rédei theorem)**.** A tournament always has at least one hamp.

**Theorem 1.1.7** (Hard Rédei theorem)**.** Let $D$ be a tournament. Then,

$$(\text{\# of hamps of } D) \text{ is odd.}$$

We shall prove these two theorems today. Clearly, the Easy Rédei Theorem follows from the Hard one, since an odd number cannot be 0. Thus, it will suffice to prove the Hard one.

First, we recall two results from Lecture 11:

**Theorem 1.1.8** (Berge's theorem)**.** Let $D = (V, A)$ be a simple digraph. Then,

$$\left(\text{\# of hamps of } \overline{D}\right) \equiv (\text{\# of hamps of } D) \bmod 2.$$

**Proposition 1.1.9.** Let $D = (V, A)$ be a simple digraph. Then,

$$\left(\text{\# of hamps of } D^{\text{rev}}\right) = (\text{\# of hamps of } D).$$

Here is our crucial lemma for the proof of the hard Rédei theorem:

[1] The word "**hamp**" is short for "Hamiltonian path". Here, we understand the 0-tuple () to be a hamp if the tournament has no vertices.

**Lemma 1.1.10.** Let $D = (V, A)$ be a tournament, and let $vw \in A$ be an arc of $D$.

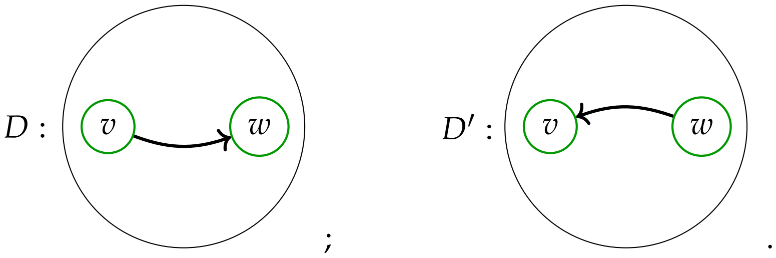
Let $D'$ be the digraph obtained from $D$ by reversing the arc $vw$. In other words, let

$$D' := (V, \ (A \setminus \{vw\}) \cup \{wv\}).$$

Then, $D'$ is again a tournament, and satisfies

$$(\text{\# of hamps of } D) \equiv (\text{\# of hamps of } D') \bmod 2.$$

Here is a visualization of the setup of Lemma 1.1.10:



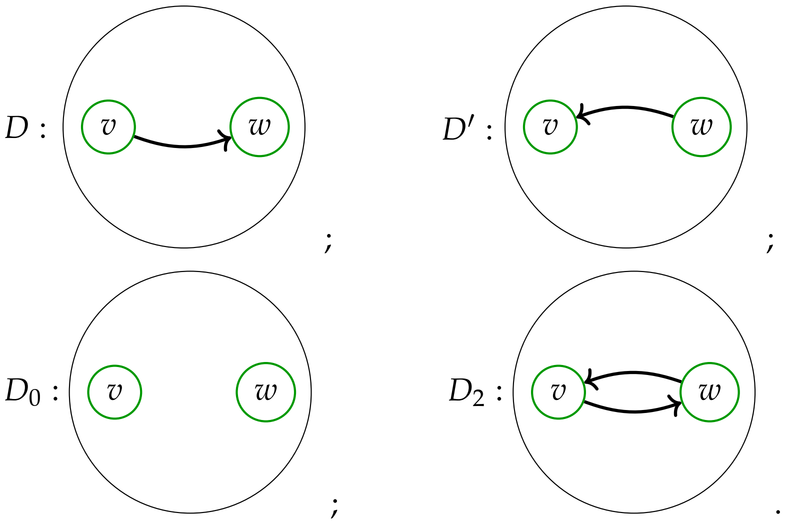

(Here, we are only showing the arcs joining $v$ with $w$, since $D$ and $D'$ agree in all other arcs.)

*Proof of Lemma 1.1.10.* First of all, $D'$ is clearly a tournament. It remains to prove the congruence.

We introduce two more digraphs: Let

$$D_0 := (\text{the digraph } D \text{ with the arc } vw \text{ removed}) \qquad \text{and}$$
$$D_2 := (\text{the digraph } D \text{ with the arc } wv \text{ added}).$$

Note that these are not tournaments any more. Here is a comparative illustration of all four digraphs $D$, $D'$, $D_0$ and $D_2$ (again showing only the arcs joining $v$ with $w$, since there are no differences in the other arcs):



The digraph $D_0$ is $D'$ with the arc $wv$ removed. Therefore, a hamp of $D_0$ is the same as a hamp of $D'$ that does not use the arc $wv$. Hence,

$$\begin{aligned}
&(\text{\# of hamps of } D_0) \\
&= \left(\text{\# of hamps of } D' \text{ that do not use the arc } wv\right) \\
&= \left(\text{\# of hamps of } D'\right) - \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right).
\end{aligned}$$

Similarly, since $D$ is $D_2$ with the arc $wv$ removed, we have

$$\begin{aligned}
&(\text{\# of hamps of } D) \\
&= (\text{\# of hamps of } D_2) - (\text{\# of hamps of } D_2 \text{ that use the arc } wv) \\
&= (\text{\# of hamps of } D_2) - \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right)
\end{aligned}$$

(the last equality is because a hamp of $D_2$ that uses the arc $wv$ cannot use the arc $vw$, and therefore is automatically a hamp of $D'$ as well, and of course the converse is obviously true).

However, from the previously proved equality

$$\begin{aligned}
&(\text{\# of hamps of } D_0) \\
&= \left(\text{\# of hamps of } D'\right) - \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\left(\text{\# of hamps of } D'\right) \\
&= (\text{\# of hamps of } D_0) + \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right) \\
&\equiv (\text{\# of hamps of } D_0) - \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right) \bmod 2
\end{aligned}$$

(since $x + y \equiv x - y \bmod 2$ for any integers $x$ and $y$). Thus, if we can show that

$$(\text{\# of hamps of } D_2) \equiv (\text{\# of hamps of } D_0) \bmod 2,$$

then we will be able to conclude that

$$\begin{aligned}
&(\text{\# of hamps of } D) \\
&= \underbrace{(\text{\# of hamps of } D_2)}_{\equiv (\text{\# of hamps of } D_0) \bmod 2} - \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right) \\
&\equiv (\text{\# of hamps of } D_0) - \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right) \\
&\equiv \left(\text{\# of hamps of } D'\right) \bmod 2,
\end{aligned}$$

and the proof of the lemma will be complete.

So let us show this. Recall that $D$ is a tournament. Thus, the non-loop arcs of $\overline{D}$ are precisely the arcs of $D^{\text{rev}}$ (by Proposition 1.1.5). Hence, the non-loop arcs of $\overline{D_0}$ are precisely the arcs of $D_2^{\text{rev}}$ (since $\overline{D_0}$ is just $\overline{D}$ with the extra arc $vw$ added, and since $D_2^{\text{rev}}$ is just $D^{\text{rev}}$ with the extra arc $vw$ added). Therefore, the digraphs $\overline{D_0}$ and $D_2^{\text{rev}}$ are equal "up to loops" (i.e., they have the same vertices and the same non-loop arcs). Since loops don't matter for hamps, these two digraphs thus have the same of hamps. Hence,

$$\left(\text{\# of hamps in } \overline{D_0}\right) = \left(\text{\# of hamps in } D_2^{\text{rev}}\right) = \left(\text{\# of hamps in } D_2\right)$$

(by Proposition 1.1.9), and therefore

$$\left(\text{\# of hamps in } D_2\right) = \left(\text{\# of hamps in } \overline{D_0}\right) \equiv \left(\text{\# of hamps in } D_0\right) \bmod 2$$

(by Theorem 1.1.8). As explained above, this completes the proof of Lemma 1.1.10. □

Now, the Hard Rédei theorem has become easy:

*Proof of Theorem 1.1.7.* We need to prove that the # of hamps of $D$ is odd. Lemma 1.1.10 tells us that the parity of this # does not change when we reverse a single arc of $D$. Thus, of course, if we reverse several arcs of $D$, then this parity does not change either. However, we can WLOG assume that the vertices of $D$ are $1, 2, \ldots, n$ for some $n \in \mathbb{N}$, and then, by reversing the appropriate arcs, we can ensure that the arcs of $D$ are

$$\begin{array}{c} 12,\ 13,\ 14,\ \ldots,\ 1n, \\ \quad 23,\ 24,\ \ldots,\ 2n, \\ \quad\quad \cdots, \\ \quad\quad\quad (n-1)\, n \end{array}$$

(i.e., each arc of $D$ has the form $ij$ with $i < j$). But at this point, the tournament $D$ has only one hamp: namely, $(1, 2, \ldots, n)$. So $(\text{\# of hamps of } D) = 1$ is odd at this point. Since the parity of the # of hamps of $D$ has not changed as we reversed our arcs, we thus conclude that it has always been odd. This proves the Hard Rédei theorem (Theorem 1.1.7). □

As we already mentioned, the Easy Rédei theorem follows from the Hard Rédei theorem. But it also has a short self-contained proof ([17s-lec7, Theorem 1.4.9]).

**Remark 1.1.11.** Theorem 1.1.7 shows that the # of hamps in a tournament is an odd positive integer. Can it be any odd positive integer, or are certain odd positive integers impossible?

Surprisingly, 7 and 21 are impossible. All other odd numbers between 1 and 80555 are possible. For higher numbers, the answer is not known so far. See MathOverflow question #232751 ([MO232751]) for more details.

## 1.2. Hamiltonian cycles in tournaments

By the Easy Rédei theorem, every tournament has a hamp. But of course, not every tournament has a hamc[2]. One obstruction is clear:

[2] Recall that "**hamc**" is our shorthand for "Hamiltonian cycle".

**Proposition 1.2.1.** If a digraph $D$ has a hamc, then $D$ is strongly connected.

In general, this is only a necessary criterion for a hamc, not a sufficient one. Not every strongly connected digraph has a hamc. However, it turns out that for tournaments, it is also sufficient, as long as the tournament has enough vertices:

**Theorem 1.2.2** (Camion's theorem)**.** If a tournament $D$ is strongly connected and has at least two vertices, then $D$ has a hamc.

*Proof sketch.* A detailed proof can be found in [17s-lec7, Theorem 1.5.5]; here is just a very rough sketch.

Let $D = (V, A)$ be a strongly connected tournament with at least two vertices.[3] We must show that $D$ has a hamc.

It is easy to see that $D$ has a cycle. Let $\mathbf{c} = (v_1, v_2, \ldots, v_k, v_1)$ be a cycle of maximum length. We shall show that $\mathbf{c}$ is a hamc.

Let $C$ be the set $\{v_1, v_2, \ldots, v_k\}$ of all vertices of this cycle $\mathbf{c}$.

A vertex $w \in V \setminus C$ will be called a **to-vertex** if there exists an arc from some $v_i$ to $w$.

A vertex $w \in V \setminus C$ will be called a **from-vertex** if there exists an arc from $w$ to some $v_i$.

Since $D$ is a tournament, each vertex in $V \setminus C$ is a to-vertex or a from-vertex. In theory, a vertex could be both (having an arc from some $v_i$ and also an arc to some other $v_j$). However, this does not actually happen. To see why, argue as follows:

- If a to-vertex $w$ has an arc from some $v_i$, then it must also have an arc from $v_{i+1}$ [4] (because otherwise there would be an arc from $w$ to $v_{i+1}$, and then we could make our cycle $\mathbf{c}$ longer by interjecting $w$ between $v_i$ and $v_{i+1}$; but this would contradict the fact that $\mathbf{c}$ is a cycle of maximum length).

- Iterating this argument, we see that if a to-vertex $w$ has an arc from some $v_i$, then it must also have an arc from $v_{i+1}$, an arc from $v_{i+2}$, an arc from $v_{i+3}$, and so on; i.e., it must have an arc from each vertex of $\mathbf{c}$. Consequently, $w$ cannot be a from-vertex. This shows that a to-vertex cannot be a from-vertex.

Let $F$ be the set of all from-vertices, and let $T$ be the set of all to-vertices. Then, as we have just shown, $F$ and $T$ are disjoint. Moreover, $F \cup T = V \setminus C$.

[3]By the way, a tournament with exactly two vertices cannot be strongly connected (as it has only 1 arc). Thus, by requiring $D$ to have at least two vertices, we have actually guaranteed that $D$ has at least three vertices.

[4]Here, indices are periodic modulo $k$, so that $v_{k+1}$ means $v_1$.

Since a to-vertex cannot be a from-vertex, we furthermore conclude that any to-vertex has an arc from each vertex of $\mathbf{c}$ (otherwise, it would be a from-vertex), and that any from-vertex has an arc to each vertex of $\mathbf{c}$ (otherwise, it would be a to-vertex).

Next, we argue that there cannot be an arc from a to-vertex $t$ to a from-vertex $f$. Indeed, if there was such an arc, then we could make the cycle $\mathbf{c}$ longer by interjecting $t$ and $f$ between (say) $v_1$ and $v_2$.

In total, we now know that every vertex of $D$ belongs to one of the three disjoint sets $C$, $F$ and $T$, and furthermore there is no arc from $T$ to $F$, no arc from $T$ to $C$, and no arc from $C$ to $F$. Thus, there exists no walk from a vertex in $T$ to a vertex in $C$ (because there is no way out of $T$). This would contradict the fact that $D$ is strongly connected, unless the set $T$ is empty. Hence, $T$ must be empty. Similarly, $F$ must be empty. Since $F \cup T = V \setminus C$, this entails that $V \setminus C$ is empty, so that $V = C$. In other words, each vertex of $D$ is on our cycle $\mathbf{c}$. Therefore, $\mathbf{c}$ is a hamc. This proves Camion's theorem. □

## 1.3. Application of tournaments to the Vandermonde determinant

To wrap up the topic of tournaments, let me briefly discuss a curious application of their theory: a combinatorial proof of the Vandermonde determinant formula. See [17s-lec8] for the many details I'll be omitting.

Recall the Vandermonde determinant formula:

**Theorem 1.3.1** (Vandermonde determinant formula)**.** Let $x_1, x_2, \ldots, x_n$ be $n$ numbers (or, more generally, elements of a commutative ring). Consider the $n \times n$-matrix

$$V := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \left(x_j^{i-1}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n}.$$

Then, its determinant is

$$\det V = \prod_{1 \leq i < j \leq n} \left(x_j - x_i\right).$$

There are many simple proofs of this theorem (e.g., a few on its ProofWiki page, which works with the transpose matrix). I will now outline a combinatorial one, using tournaments. This proof goes back to Ira Gessel's 1979 paper [Gessel79].

First, how do $\det V$ and $\prod\limits_{1\leq i<j\leq n} (x_i - x_j)$ relate to tournaments?

As a warmup, let's assume that we have some number $y_{(i,j)}$ given for each pair $(i,j)$ of integers, and let's expand the product

$$\left(y_{(1,2)} + y_{(2,1)}\right) \left(y_{(1,3)} + y_{(3,1)}\right) \left(y_{(2,3)} + y_{(3,2)}\right).$$

The result is a sum of 8 products, one for each way to pluck an addend out of each of the three little sums:

$$\begin{aligned}
&\left(y_{(1,2)} + y_{(2,1)}\right) \left(y_{(1,3)} + y_{(3,1)}\right) \left(y_{(2,3)} + y_{(3,2)}\right) \\
&= y_{(1,2)}y_{(1,3)}y_{(2,3)} + y_{(1,2)}y_{(1,3)}y_{(3,2)} + y_{(1,2)}y_{(3,1)}y_{(2,3)} + y_{(1,2)}y_{(3,1)}y_{(3,2)} \\
&\qquad + y_{(2,1)}y_{(1,3)}y_{(2,3)} + y_{(2,1)}y_{(1,3)}y_{(3,2)} + y_{(2,1)}y_{(3,1)}y_{(2,3)} + y_{(2,1)}y_{(3,1)}y_{(3,2)}.
\end{aligned}$$

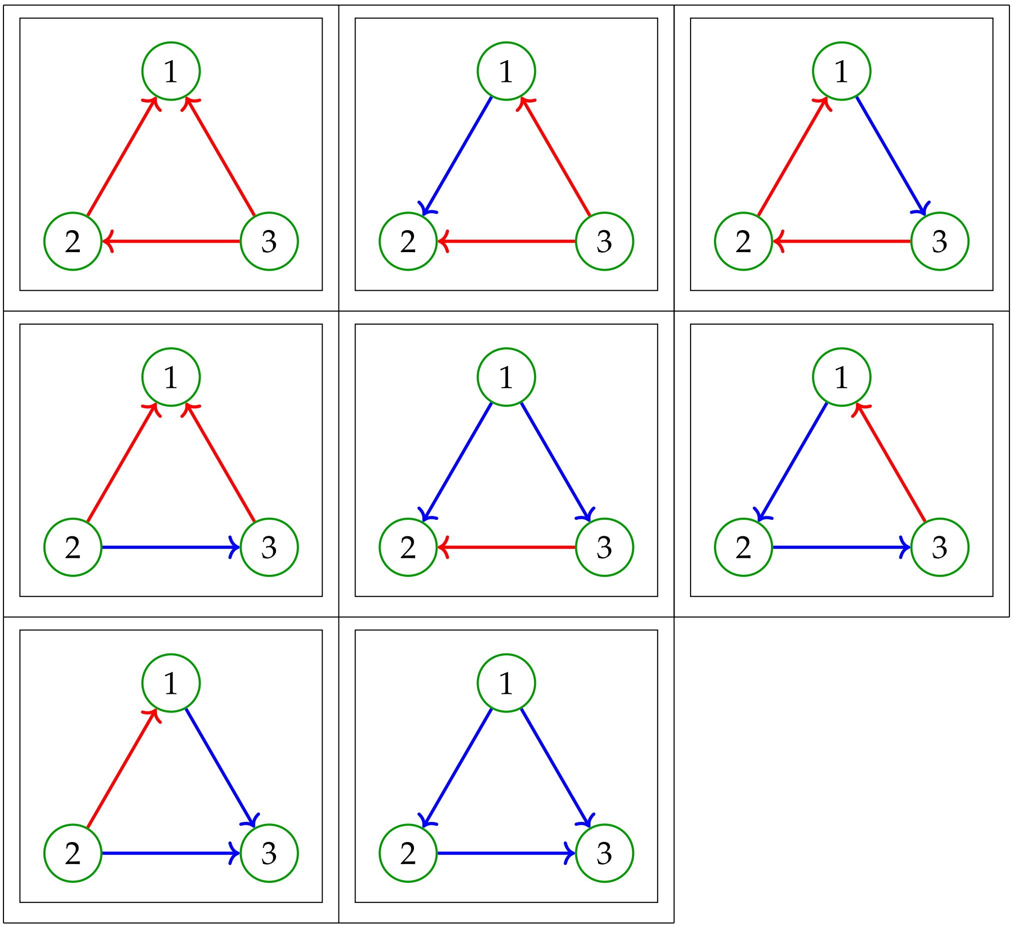
Note that each of the 8 products obtained has the form $y_a y_b y_c$, where

- $a$ is one of the pairs $(1,2)$ and $(2,1)$,
- $b$ is one of the pairs $(1,3)$ and $(3,1)$, and
- $c$ is one of the pairs $(2,3)$ and $(3,2)$.

We can view these pairs $a$, $b$ and $c$ as the arcs of a tournament with vertex set $\{1,2,3\}$. Thus, our above expansion can be rewritten more compactly as follows:

$$\begin{aligned}
&\left(y_{(1,2)} + y_{(2,1)}\right) \left(y_{(1,3)} + y_{(3,1)}\right) \left(y_{(2,3)} + y_{(3,2)}\right) \\
&= \sum_{\substack{D \text{ is a tournament} \\ \text{with vertex set } \{1,2,3\}}} \ \prod_{(i,j) \text{ is an arc of } D} y_{(i,j)}.
\end{aligned}$$

For reference, here are all the 8 tournaments with vertex set $\{1, 2, 3\}$:




Here, for convenience, we are drawing an arc $ij$ in blue if $i < j$ and in red otherwise.

This expansion can be generalized: We have

$$\prod_{1 \leq i < j \leq n} \left( y_{(i,j)} + y_{(j,i)} \right) = \sum_{\substack{D \text{ is a tournament} \\ \text{with vertex set } \{1,2,\ldots,n\}}} \prod_{(i,j) \text{ is an arc of } D} y_{(i,j)}.$$

Substituting $y_{(i,j)} = \begin{cases} x_j, & \text{if } i < j; \\ -x_j, & \text{if } i \geq j \end{cases}$ in this equality, we obtain

$$\prod_{1\leq i<j\leq n} (x_j - x_i) = \sum_{\substack{D \text{ is a tournament} \\ \text{with vertex set } \{1,2,\ldots,n\}}} \underbrace{\prod_{(i,j) \text{ is an arc of } D} \begin{cases} x_j, & \text{if } i < j; \\ -x_j, & \text{if } i \geq j \end{cases}}_{\substack{=(-1)^{(\# \text{ of red arcs of } D)} \prod_{j=1}^{n} x_j^{\deg^- j} \\ \text{(where } \deg^- j \text{ means the indegree of } j \text{ in } D, \\ \text{and where the ``red arcs'' are the arcs } ij \text{ with } i>j)}}$$

$$= \sum_{\substack{D \text{ is a tournament} \\ \text{with vertex set } \{1,2,\ldots,n\}}} (-1)^{(\# \text{ of red arcs of } D)} \prod_{j=1}^{n} x_j^{\deg^- j}.$$

We shall refer to this sum as the "big sum".

On the other hand, if we let $S_n$ be the group of permutations of $\{1, 2, \ldots, n\}$, and if we denote the sign of a permutation $\sigma$ by $\operatorname{sign} \sigma$, then we have

$$\det V = \det \left( V^T \right) = \sum_{\sigma \in S_n} \operatorname{sign} \sigma \cdot \prod_{j=1}^{n} x_j^{\sigma(j)-1}$$

(by the definition of a determinant). We shall refer to this sum as the "small sum".

Our goal is to prove that the big sum equals the small sum. To prove this, we must verify the following:

1. Each addend of the small sum is an addend of the big sum. Indeed, for each permutation $\sigma \in S_n$, there is a certain tournament $T_\sigma$ that has

   $$(-1)^{(\# \text{ of red arcs of } T_\sigma)} \prod_{j=1}^{n} x_j^{\deg^- j} = \operatorname{sign} \sigma \cdot \prod_{j=1}^{n} x_j^{\sigma(j)-1}.$$

   Can you find this $T_\sigma$ ?

2. All the addends of the big sum that are **not** addends of the small sum cancel each other out. Why?

   The basic idea is to argue that if a tournament $D$ appears in the big sum but not in the small sum, then $D$ has a 3-cycle (i.e., a cycle of length 3). When we reverse such a 3-cycle (i.e., we reverse each of its arcs), the indegrees of all vertices are preserved, but the sign $(-1)^{(\# \text{ of red arcs of } D)}$ is flipped (since three arcs change their orientation).

   This suffices to show that for each addend that appears in the big sum but not in the small sum, there is another addend with the same magnitude but with opposite sign. Unfortunately, this in itself does not suffice to ensure that all these addends cancel out; for example, the sum $1+1+1+(-1)$ has the same property but does not equal $0$. We need to show that the # of addends with positive sign (i.e., with $(-1)^{(\text{\# of red arcs of } D)} = 1$) and a given magnitude equals the # of addends with negative sign (i.e., with $(-1)^{(\text{\# of red arcs of } D)} = -1$) and the same magnitude.

One way to achieve this would be by constructing a bijection (aka "perfect matching") between the "positive" and the "negative" addends. This is tricky here: We would have to decide **which** 3-cycle to reverse (as there are usually many of them), and this has to be done in a bijective way (so that two "positive" addends don't get assigned the same "negative" partner).

A less direct, but easier way is the following: Fix a positive integer $k$, and consider only the tournaments with exactly $k$ many 3-cycles. For each such tournament, we can reverse any of its $k$ many 3-cycles. It can be shown (nice exercise!) that reversing the arcs of a 3-cycle does not change the # of all 3-cycles; thus, we don't accidentally change our $k$ in the process. Thus, we find a "$k$-to-$k$" correspondence between the "positive" addends of a given magnitude and the "negative" addends of the same magnitude. As one can easily see, this entails that the former and the latter are equinumerous, and thus really cancel out. The addends that remain are exactly those in the small sum.

As already mentioned, this is only a rough summary of the proof; the details can be found in [17s-lec8].

## References

[17s-lec7] Darij Grinberg, *UMN, Spring 2017, Math 5707: Lecture 7 (Hamiltonian paths in digraphs)*, 14 May 2022.
`https://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec7.pdf`

[17s-lec8] Darij Grinberg, *UMN, Spring 2017, Math 5707: Lecture 8 (Vandermonde determinant using tournaments)*, 14 May 2022.
`https://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec8.pdf`

[Gessel79] Ira Gessel, *Tournaments and Vandermonde's Determinant*, Journal of Graph Theory **3** (1979), pp. 305–307.

[MO232751] bof and Gordon Royle, *MathOverflow question #232751 ("The number of Hamiltonian paths in a tournament")*.
`https://mathoverflow.net/questions/232751/the-number-of-hamiltonian-paths-in-a-tournament`