Math 530 Spring 2022, Lecture 11: tournaments and Hamiltonian paths

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

See Lecture 7 from my Spring 2017 course (`https://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec7.pdf`) for today's material in more detail.

1. Digraphs and multidigraphs (cont'd)

Last time, we defined hamps (Hamiltonian paths) for simple digraphs. Today, we shall study them further. First, we introduce two more operations on simple digraphs.

1.1. The reverse and complement digraphs

Definition 1.1.1. Let $D = (V, A)$ be a simple digraph. Then:

(a) The elements of $(V \times V) \setminus A$ will be called the **non-arcs** of D.

(b) The **reversal** of a pair $(i, j) \in V \times V$ means the pair (j, i).

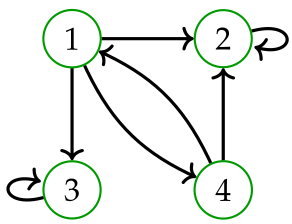
(c) We define D^{rev} as the simple digraph (V, A^{rev}), where

$$A^{\text{rev}} = \{(j, i) \mid (i, j) \in A\}.$$

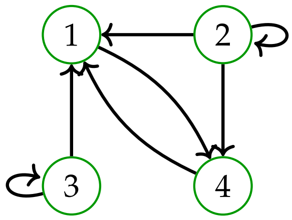
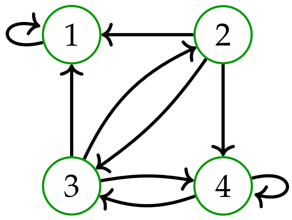
Thus, D^{rev} is the digraph obtained from D by reversing each arc (i.e., swapping its source and its target). This is called the **reversal** of D.

(d) We define $\overline{D}$ as the simple digraph $(V, \ (V \times V) \setminus A)$. This is the digraph that has the same vertices as D, but whose arcs are precisely the non-arcs of D. This digraph $\overline{D}$ is called the **complement** of D.

Example 1.1.2. Let

$D =$  .

Then,

$D^{\text{rev}} =$  and $\overline{D} =$  .

Convention 1.1.3. In the following, the symbol # means "number". For example,

$$(\text{\# of subsets of } \{1, 2, 3\}) = 8.$$

We now shall try to count hamps in simple digraphs. As a warmup, here is a particularly simple case:

Proposition 1.1.4. Let D be the simple digraph (V, A), where

$$V = \{1, 2, \ldots, n\} \qquad \text{for some } n \in \mathbb{N},$$

and where

$$A = \{(i, j) \mid i < j\}.$$

Then, $(\text{\# of hamps of } D) = 1$.

Proof. It is easy to see that the only hamp of D is $(1, 2, \ldots, n)$. □

Proposition 1.1.5. Let D be a simple digraph. Then,

$$(\text{\# of hamps of } D^{\text{rev}}) = (\text{\# of hamps of } D).$$

Proof. The hamps of D^{rev} are obtained from the hamps of D by walking backwards. □

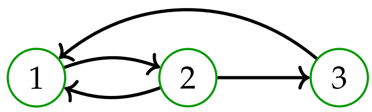
So far, so boring. What about this:

Theorem 1.1.6 (Berge)**.** Let D be a simple digraph. Then,

$$\left(\text{\# of hamps of } \overline{D}\right) \equiv (\text{\# of hamps of } D) \bmod 2.$$

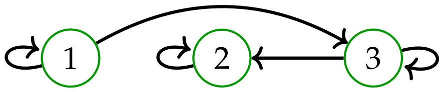
This is much less obvious or even expected. We first give an example:

Example 1.1.7. Let D be the following digraph:

$D =$  .

This digraph has 3 hamps: $(1,2,3)$ and $(2,3,1)$ and $(3,1,2)$.

Its complement $\overline{D}$ looks as follows:

$\overline{D} =$  .

It has only 1 hamp: $(1,3,2)$.

Thus, in this case, Theorem 1.1.6 says that $1 \equiv 3 \mod 2$.

Proof of Theorem 1.1.6. (This is an outline; see [17s-lec7, proof of Theorem 1.3.6] for more details.)

Write the simple digraph D as $D = (V, A)$, and assume WLOG that $V \neq \varnothing$. Set $n = |V|$.

A V**-listing** will mean a list of elements of V that contains each element of V exactly once. (Thus, each V-listing is an n-tuple, and there are $n!$ many V-listings.) Note that a V-listing is the same as a hamp of the "complete" digraph $(V, V \times V)$. Any hamp of D or of $\overline{D}$ is therefore a V-listing, but not every V-listing is a hamp of D or $\overline{D}$.

If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a V-listing, then we define a set

$$P(\sigma) := \{\sigma_1\sigma_2,\ \sigma_2\sigma_3,\ \ldots,\ \sigma_{n-1}\sigma_n\}.$$

We call this set $P(\sigma)$ the **arc set** of σ. When we regard σ as a hamp of $(V, V \times V)$, this set $P(\sigma)$ is just the set of all arcs of σ. Note that this is an $(n-1)$-element set. We make a few easy observations (prove them!):

> *Observation 1:* We can reconstruct a V-listing σ from its arc set $P(\sigma)$. In other words, the map $\sigma \mapsto P(\sigma)$ is injective.
>
> *Observation 2:* Let σ be a V-listing. Then, σ is a hamp of D if and only if $P(\sigma) \subseteq A$.
>
> *Observation 3:* Let σ be a V-listing. Then, σ is a hamp of $\overline{D}$ if and only if $P(\sigma) \subseteq (V \times V) \setminus A$.

Now, let N be the # of pairs (σ, B), where σ is a V-listing and B is a subset of A satisfying $B \subseteq P(\sigma)$. Thus,

$$N = \sum_{\sigma \text{ is a } V\text{-listing}} N_\sigma,$$

where

$$N_\sigma = (\text{\# of subsets } B \text{ of } A \text{ satisfying } B \subseteq P(\sigma)).$$

But we also have

$$N = \sum_{B \text{ is a subset of } A} N^B,$$

where

$$N^B = (\text{\# of } V\text{-listings } \sigma \text{ satisfying } B \subseteq P(\sigma)).$$

Let us now relate these two sums to hamps. We begin with $\sum\limits_{\sigma \text{ is a } V\text{-listing}} N_\sigma$.

We shall use the **Iverson bracket notation**: i.e., the notation $[\mathcal{A}]$ for the truth value of a statement $\mathcal{A}$. This truth value is defined to be the number 1 if $\mathcal{A}$ is true, and 0 if $\mathcal{A}$ is false. For instance,

$$[2+2=4] = 1 \qquad \text{and} \qquad [2+2=5] = 0.$$

For any V-listing σ, we have

$$\begin{aligned}
N_\sigma &= (\text{\# of subsets } B \text{ of } A \text{ satisfying } B \subseteq P(\sigma)) \\
&= (\text{\# of subsets } B \text{ of } A \cap P(\sigma)) \\
&= 2^{|A \cap P(\sigma)|} \\
&\equiv [|A \cap P(\sigma)| = 0] \qquad (\text{since } 2^m \equiv [m=0] \bmod 2 \text{ for each } m \in \mathbb{N}) \\
&= [A \cap P(\sigma) = \varnothing] \qquad \left(\begin{array}{c}\text{since equivalent statements have the} \\ \text{same truth value}\end{array}\right) \\
&= [P(\sigma) \subseteq (V \times V) \setminus A] \qquad (\text{since } P(\sigma) \text{ is always a subset of } V \times V) \\
&= \left[\sigma \text{ is a hamp of } \overline{D}\right] \bmod 2 \qquad (\text{by Observation 3}).
\end{aligned}$$

So

$$\begin{aligned}
N &= \sum_{\sigma \text{ is a } V\text{-listing}} \underbrace{N_\sigma}_{\equiv\left[\sigma \text{ is a hamp of } \overline{D}\right] \bmod 2} \\
&\equiv \sum_{\sigma \text{ is a } V\text{-listing}} \left[\sigma \text{ is a hamp of } \overline{D}\right] \\
&= \left(\text{\# of } V\text{-listings } \sigma \text{ that are hamps of } \overline{D}\right) \\
&\qquad \left(\begin{array}{c}\text{because } \sum\limits_{\sigma \text{ is a } V\text{-listing}} \left[\sigma \text{ is a hamp of } \overline{D}\right] \text{ is a sum} \\ \text{of several 1's and several 0's, and the 1's in this} \\ \text{sum correspond precisely to} \\ \text{the } V\text{-listings } \sigma \text{ that are hamps of } \overline{D}\end{array}\right) \\
&= \left(\text{\# of hamps of } \overline{D}\right) \bmod 2.
\end{aligned}$$

What about the other expression for N ? Recall that

$$N = \sum_{B \text{ is a subset of } A} N^B,$$

where

$$N^B = (\text{\# of } V\text{-listings } \sigma \text{ satisfying } B \subseteq P(\sigma)).$$

We want to prove that this sum equals (# of hamps of D), at least modulo 2.

So let B be a subset of A. We want to know $N^B \bmod 2$. In other words, we want to know when N^B is odd.

Let us first assume that N^B is odd, and see what follows from this.

Since N^B is odd, we have $N^B > 0$. Thus, there exists **at least one** V-listing σ satisfying $B \subseteq P(\sigma)$. We shall now draw some conclusions from this.

First, a definition: A **path cover** of V means a set of paths in the "complete" digraph $(V, V \times V)$ such that each vertex $v \in V$ is contained in exactly one of these paths. The **set of arcs** of such a path cover is simply the set of all arcs of all its paths. For example, if $V = \{1,2,3,4,5,6,7\}$, then

$$\{(1,3,5), \ (2), \ (6), \ (7,4)\}$$

is a path cover of V, and its set of arcs is $\{13, \ 35, \ 74\}$.

Now, ponder the following: If we remove an arc $v_i v_{i+1}$ from a path $(v_1, v_2, \ldots, v_k)$, then this path breaks up into two paths $(v_1, v_2, \ldots, v_i)$ and $(v_{i+1}, v_{i+2}, \ldots, v_k)$. Thus, if we remove some arcs from the arc set $P(\sigma)$ of a V-listing σ, then we obtain the set of arcs of a path cover of V. (For instance, removing the arcs 52, 26 and 67 from the arc set $P(\sigma)$ of the V-listing $\sigma = (1,3,5,2,6,7,4)$ yields precisely the path cover $\{(1,3,5), \ (2), \ (6), \ (7,4)\}$ that we just showed as an example.)

Now, recall that there exists **at least one** V-listing σ satisfying $B \subseteq P(\sigma)$. Hence, B is obtained by removing some arcs from the arc set $P(\sigma)$ of this V-listing σ. Therefore, B is the set of arcs of a path cover of V (by the claim of the preceding paragraph). Let us say that this path cover consists of exactly r paths. Then,

$$(\text{\# of } V\text{-listings } \sigma \text{ satisfying } B \subseteq P(\sigma)) = r!,$$

because any such V-listing σ can be constructed by concatenating the r paths in our path cover in some order (and there are $r!$ possible orders).

Thus, $N^B = (\text{\# of } V\text{-listings } \sigma \text{ satisfying } B \subseteq P(\sigma)) = r!$. But we have assumed that N^B is odd. So $r!$ is odd. Since r is positive (because $V \neq \varnothing$, so our path cover must contain at least one path), this entails that $r = 1$. So our path cover is just a single path; this path is a path of D (since its set of arcs B is a subset of A) and therefore is a hamp of D (since it constitutes a path cover of V all by itself). If we denote it by σ, then we have $B = P(\sigma)$ (since B is the set of arcs of the path cover that consists of σ alone).

Forget our assumption that N^B is odd. We have thus shown that if N^B is odd, then $B = P(\sigma)$ for some hamp σ of D.

Conversely, it is easy to see that if $B = P(\sigma)$ for some hamp σ of D, then N^B is odd (and actually equals 1).

Combining these two results, we see that N^B is odd **if and only if** $B = P(\sigma)$ for some hamp σ of D. Therefore,

$$\left[N^B \text{ is odd}\right] = [B = P(\sigma) \text{ for some hamp } \sigma \text{ of } D].$$

However,

$$\begin{aligned} N^B &\equiv \left[N^B \text{ is odd}\right] \qquad \left(\text{since } m \equiv [m \text{ is odd}] \bmod 2 \text{ for any } m \in \mathbb{Z}\right) \\ &= [B = P(\sigma) \text{ for some hamp } \sigma \text{ of } D] \bmod 2. \end{aligned}$$

We have proved this congruence for every subset B of A. Thus,

$$\begin{aligned} N &= \sum_{B \text{ is a subset of } A} \underbrace{N^B}_{\equiv [B = P(\sigma) \text{ for some hamp } \sigma \text{ of } D] \bmod 2} \\ &\equiv \sum_{B \text{ is a subset of } A} [B = P(\sigma) \text{ for some hamp } \sigma \text{ of } D] \\ &= (\# \text{ of subsets } B \text{ of } A \text{ such that } B = P(\sigma) \text{ for some hamp } \sigma \text{ of } D) \\ &= (\# \text{ of sets of the form } P(\sigma) \text{ for some hamp } \sigma \text{ of } D) \\ &\qquad \left(\begin{array}{c} \text{because each set of the form } P(\sigma) \text{ for some} \\ \text{hamp } \sigma \text{ of } D \text{ is a subset of } A \text{ (by Observation 2)} \end{array}\right) \\ &= (\# \text{ of hamps of } D) \bmod 2 \end{aligned}$$

(indeed, Observation 1 shows that different hamps σ have different sets $P(\sigma)$, so counting the sets $P(\sigma)$ for all hamps σ is equivalent to counting the hamps σ themselves).

Now we have proved that $N \equiv \left(\# \text{ of hamps of } \overline{D}\right) \bmod 2$ and $N \equiv (\# \text{ of hamps of } D) \bmod 2$. Comparing these two congruences, we obtain

$$\left(\# \text{ of hamps of } \overline{D}\right) \equiv (\# \text{ of hamps of } D) \bmod 2.$$

This proves Berge's theorem. □

1.2. Tournaments

We now introduce a special class of simple digraphs.

Definition 1.2.1. A digraph D is said to be **loopless** if it has no loops.

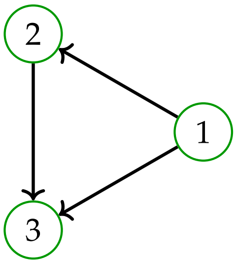
Definition 1.2.2. A **tournament** is defined to be a loopless simple digraph D that satisfies the

- **Tournament axiom:** For any two distinct vertices u and v of D, **exactly** one of (u, v) and (v, u) is an arc of D.

Equivalently:

Proposition 1.2.3. A simple digraph D is a tournament if and only if D^{rev} is $\overline{D}$ without the loops.

Example 1.2.4. The following digraph is a tournament:



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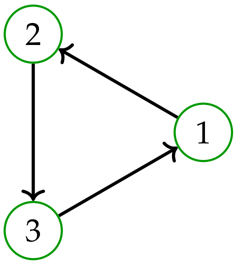
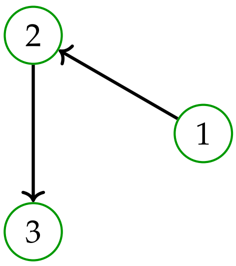
The following digraph is a tournament as well:



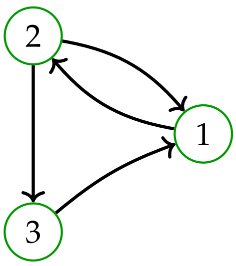
.

However, the following digraph is not a tournament:



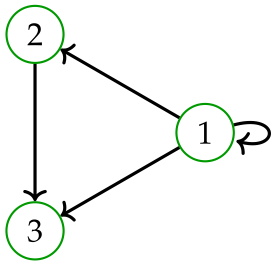
,

because the tournament axiom is not satisfied for $u = 1$ and $v = 3$. Nor is the following digraph a tournament:



,

because the tournament axiom is not satisfied for $u = 1$ and $v = 2$. Finally, the digraph



is not a tournament either, since it is not loopless.

The digraph D in Proposition 1.1.4 always is a tournament.

Which tournaments have hamps? The answer is surprisingly simple:

Theorem 1.2.5 (Easy Rédei theorem)**.** A tournament always has at least one hamp.

Even better, and perhaps even more surprisingly:

Theorem 1.2.6 (Hard Rédei theorem)**.** Let D be a tournament. Then,

$$(\text{\# of hamps of } D) \ \text{ is odd.}$$

I will prove both theorems in the next lecture.

References

[17s-lec7] Darij Grinberg, *UMN, Spring 2017, Math 5707: Lecture 7 (Hamiltonian paths in digraphs)*, 14 May 2022.
`https://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec7.pdf`