Math 530 Spring 2022, Lecture 10: digraphs

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Digraphs and multidigraphs (cont'd)

1.1. Conversions (cont'd)

1.1.1. Multigraphs to multidigraphs

Last time, we explained how to turn any multidigraph D into a multigraph D^{und} by forgetting the directions of the arcs.

Conversely, we can turn a multigraph G into a multidigraph G^{bidir} by "duplicating" each edge (more precisely: turning each edge into two arcs with opposite orientations). Here is a formal definition:

Definition 1.1.1. Let $G = (V, E, \varphi)$ be a multigraph. For each edge $e \in E$, let us choose one of the endpoints of e and call it s_e; the other endpoint will then be called t_e. (If e is a loop, then we understand t_e to mean s_e.)

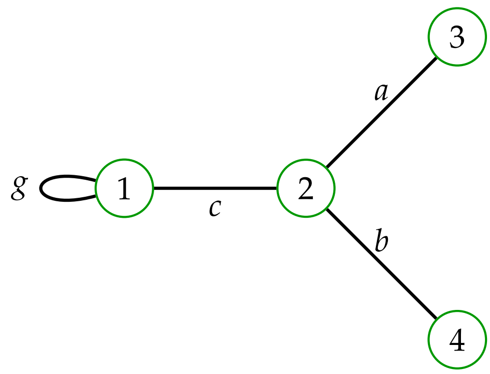
We then define G^{bidir} to be the multidigraph $(V,\ E \times \{1,2\},\ \psi)$, where the map $\psi : E \times \{1,2\} \to V \times V$ is defined as follows: For each edge $e \in E$, we set

$$\psi(e,1) = (s_e, t_e) \qquad \text{and} \qquad \psi(e,2) = (t_e, s_e).$$

We call G^{bidir} the **bidirectionalized multidigraph** of G.

Note that the map ψ depends on our choice of s_e's (that is, it depends on which endpoint of an edge e we choose to be s_e). This makes the definition of G^{bidir} non-canonical; I don't know if there is a good way to fix this. Fortunately, all choices of s_e's will lead to mutually isomorphic multidigraphs G^{bidir}. (The notion of **isomorphism** for multidigraphs is exactly the one that you expect.)

Example 1.1.2. If

$G =$ 

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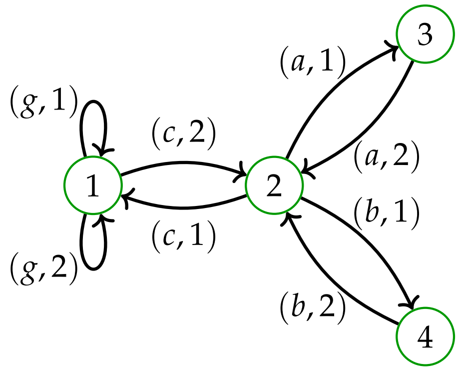
then

$G^{\text{bidir}} =$ 

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(Here, for example, we have chosen s_a to be 2, so that $t_a = 3$ and $\psi(a, 1) = (2, 3)$ and $\psi(a, 2) = (3, 2)$.) Yes, even the loops of G are duplicated in G^{bidir} !

The operation that assigns a multidigraph G^{bidir} to a multigraph G is injective – i.e., the original graph G can be uniquely reconstructed from G^{bidir}. This is in stark difference to the operation $D \mapsto D^{\text{und}}$, which destroys information (the directions of the arcs). Note that the multigraph $\left(G^{\text{bidir}}\right)^{\text{und}}$ is not isomorphic to G, since each edge of G is doubled in $\left(G^{\text{bidir}}\right)^{\text{und}}$.

1.1.2. Simple digraphs to multidigraphs

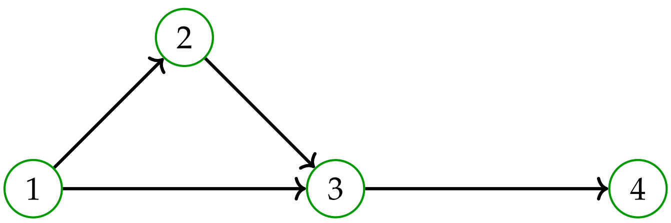
Next, we introduce another operation: one that turns simple digraphs into multidigraphs. This is very similar to the operation $G \mapsto G^{\text{mult}}$ that turns simple graphs into multigraphs, so we will even use the same notation for it. Its definition is as follows:

Definition 1.1.3. Let $D = (V, A)$ be a simple digraph. Then, the **corresponding multidigraph** D^{mult} is defined to be the multidigraph

$$(V, A, \iota),$$

where $\iota : A \to V \times V$ is the map sending each $a \in A$ to a itself.

Example 1.1.4. If

$D =$ 

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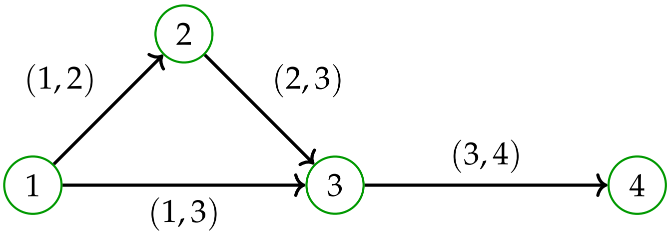
then

$D^{\mathrm{mult}} =$ 

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1.1.3. Multidigraphs to simple digraphs

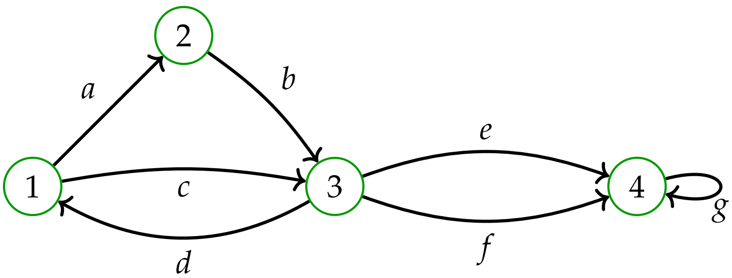
There is also an operation $D \mapsto D^{\mathrm{simp}}$ that turns multidigraphs into simple digraphs:[1]

Definition 1.1.5. Let $D = (V, A, \psi)$ be a multidigraph. Then, the **underlying simple digraph** D^{simp} of D means the simple digraph

$$(V, \ \{\psi(a) \ | \ a \in A\}) .$$

In other words, it is the simple digraph with vertex set V in which there is an arc from u to v if there exists an arc from u to v in D. Thus, D^{simp} is obtained from D by "collapsing" parallel arcs (i.e., arcs having the same source and the same target) to a single arc.

Example 1.1.6. If

$D =$ 

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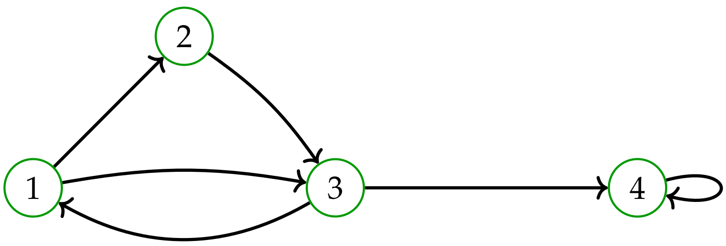
then

$D^{\mathrm{simp}} =$ 

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[1] I will use a notation that I probably should have introduced before: If u and v are two vertices of a digraph, then an "**arc from** u **to** v" means an arc with source u and target v.

Note that the arcs c and d have not been "collapsed" into one arc, since they do not have the same source and the same target. Likewise, the loop g has been preserved (unlike for undirected graphs).

1.1.4. Multidigraphs as a big tent

A takeaway from this all is that multidigraphs are the "most general" notion of graphs we have introduced so far. Indeed, using the operations we have seen so far, we can convert every notion of graphs into a multidigraph:

- Each simple graph becomes a multigraph via the $G \mapsto G^{\text{mult}}$ operation.
- Each multigraph, in turn, becomes a multidigraph via the $G \mapsto G^{\text{bidir}}$ operation.
- Each simple digraph becomes a multidigraph via the $D \mapsto D^{\text{mult}}$ operation.

Since all three of these operations are injective (i.e., lose no information), we thus can encode each of our four notions of graphs as a multidigraph. Consequently, any theorem about multidigraphs can be specialized to the other three types of graphs. This doesn't mean that any theorem on any other type of graphs can be generalized to multidigraphs, though (e.g., Mantel's theorem holds only for simple graphs) – but when it can, we will try to state it at the most general level possible, to avoid doing the same work twice.

1.2. Walks, paths, closed walks, cycles

Let us now define various kinds of walks for simple digraphs and for multidigraphs.

For simple digraphs, we imitate the definitions from Lectures 3 and 4 as best as we can, making sure to require all arcs to be traversed in the correct direction:

Definition 1.2.1. Let D be a simple digraph. Then:

(a) A **walk** (in D) means a finite sequence $(v_0, v_1, \ldots, v_k)$ of vertices of D (with $k \geq 0$) such that all of the pairs v_0v_1, v_1v_2, v_2v_3, $\ldots$, $v_{k-1}v_k$ are arcs of D. (The latter condition is vacuously true if $k = 0$.)

(b) If $\mathbf{w} = (v_0, v_1, \ldots, v_k)$ is a walk in D, then:

- The **vertices** of $\mathbf{w}$ are defined to be $v_0, v_1, \ldots, v_k$.
- The **arcs** of $\mathbf{w}$ are defined to be the pairs v_0v_1, v_1v_2, v_2v_3, $\ldots$, $v_{k-1}v_k$.

- The nonnegative integer k is called the **length** of $\mathbf{w}$. (This is the number of all arcs of $\mathbf{w}$, counted with multiplicity. It is 1 smaller than the number of all vertices of $\mathbf{w}$, counted with multiplicity.)
- The vertex v_0 is called the **starting point** of $\mathbf{w}$. We say that **w starts** (or **begins**) at v_0.
- The vertex v_k is called the **ending point** of $\mathbf{w}$. We say that **w ends** at v_k.

(c) A **path** (in D) means a walk (in D) whose vertices are distinct. In other words, a path means a walk $(v_0, v_1, \ldots, v_k)$ such that $v_0, v_1, \ldots, v_k$ are distinct.

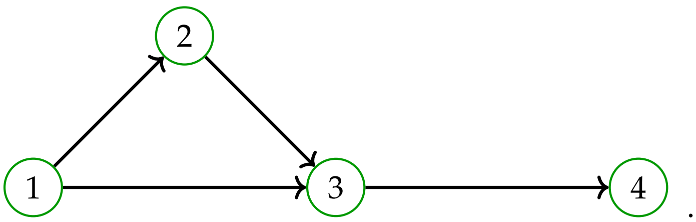
(d) Let p and q be two vertices of D. A **walk from** p **to** q means a walk that starts at p and ends at q. A **path from** p **to** q means a path that starts at p and ends at q.

(e) A **closed walk** of D means a walk whose first vertex is identical with its last vertex. In other words, it means a walk $(w_0, w_1, \ldots, w_k)$ with $w_0 = w_k$. Sometimes, closed walks are also known as **circuits** (but many authors use this latter word for something slightly different).

(f) A **cycle** of D means a closed walk $(w_0, w_1, \ldots, w_k)$ such that $k \geq 1$ and such that the vertices $w_0, w_1, \ldots, w_{k-1}$ are distinct.

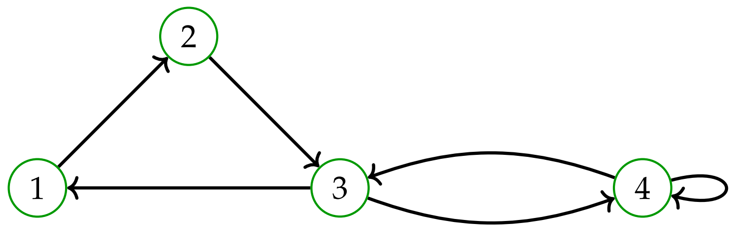
Note that we replaced the condition $k \geq 3$ by $k \geq 1$ in the definition of a cycle, since simple digraphs can have loops. Fortunately, with the arcs being directed, we no longer have to worry about the same arc being traversed back and forth, so we need no extra condition to rule this out.

Example 1.2.2. Consider the simple digraph

$D =$ 

Then, $(1, 2, 3, 4)$ and $(1, 3, 4)$ are two walks of D, and these walks are paths. But $(2, 3, 1)$ is not a walk (since you cannot use the arc 13 to get from 3 to 1). This digraph D has no cycles, and its only closed walks have length 0.

Example 1.2.3. Consider the simple digraph

$D =$ 

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Then, $(1,2,3,1)$ and $(3,4,3)$ and $(4,4)$ are cycles of D. Moreover, $(1,2,3,4,3,1)$ is a closed walk but not a cycle.

Now let's define the same concepts for multidigraphs, by modifying the analogous definitions for multigraphs we saw in Lecture 7:

Definition 1.2.4. Let $D = (V, A, \psi)$ be a multidigraph. Then:

(a) A **walk** in D means a list of the form

$$(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k) \qquad (\text{with } k \geq 0),$$

where $v_0, v_1, \ldots, v_k$ are vertices of D, where $a_1, a_2, \ldots, a_k$ are arcs of D, and where each $i \in \{1, 2, \ldots, k\}$ satisfies

$$\psi(a_i) = (v_{i-1}, v_i)$$

(that is, each arc a_i has source v_{i-1} and target v_i). Note that we have to record both the vertices **and** the arcs in our walk, since we want the walk to "know" which arcs it traverses.

The **vertices** of a walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ are $v_0, v_1, \ldots, v_k$; the **arcs** of this walk are $a_1, a_2, \ldots, a_k$. This walk is said to **start** at v_0 and **end** at v_k; it is also said to be a **walk from** v_0 **to** v_k. Its **starting point** is v_0, and its **ending point** is v_k. Its **length** is k.

(b) A **path** means a walk whose vertices are distinct.

(c) A **closed walk** (or **circuit**) means a walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ with $v_k = v_0$.

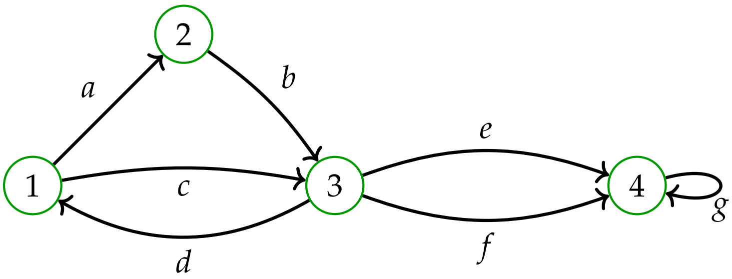
(d) A **cycle** means a closed walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ such that

- the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct;
- we have $k \geq 1$.

(This automatically implies that the arcs $a_1, a_2, \ldots, a_k$ are distinct, since each arc a_i has source v_{i-1}.)

Example 1.2.5. Consider the multidigraph

$D =$



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Then, $(1, a, 2, b, 3, d, 1)$ and $(3, d, 1, c, 3)$ and $(4, g, 4)$ are three cycles of D, whereas $(3, d, 1, a, 2, b, 3, d, 1, c, 3)$ is a circuit but not a cycle.

Now, let us see which properties of walks, paths, closed walks and cycles remain valid for digraphs.

In Lecture 3, we saw how two walks in a simple graph could be combined ("spliced together") if the ending point of the first is the starting point of the second. In Lecture 8, we generalized this to multigraphs. The same holds for multidigraphs:

Proposition 1.2.6. Let D be a multidigraph. Let u, v and w be three vertices of D. Let $\mathbf{a} = (a_0, e_1, a_1, \ldots, e_k, a_k)$ be a walk from u to v. Let $\mathbf{b} = (b_0, f_1, b_1, \ldots, f_\ell, b_\ell)$ be a walk from v to w. Then,

$$\begin{aligned} & (a_0, e_1, a_1, \ldots, e_k, a_k, f_1, b_1, f_2, b_2, \ldots, f_\ell, b_\ell) \\ & = (a_0, e_1, a_1, \ldots, a_{k-1}, e_k, b_0, f_1, b_1, \ldots, f_\ell, b_\ell) \\ & = (a_0, e_1, a_1, \ldots, a_{k-1}, e_k, v, f_1, b_1, \ldots, f_\ell, b_\ell) \end{aligned}$$

is a walk from u to w. This walk shall be denoted $\mathbf{a} * \mathbf{b}$.

Proof. The same (trivial) argument as for undirected graphs works here. □

However, unlike for undirected graphs, we can no longer reverse walks or paths. Thus, it often happens that there is a walk from u to v, but no walk from v to u.

Reducing a walk to a path (as we did in Lecture 3 for simple graphs and in Lecture 8 for multigraphs) still works for multidigraphs:

Proposition 1.2.7. Let D be a multidigraph. Let u and v be two vertices of D. Let $\mathbf{a}$ be a walk from u to v. Let k be the length of $\mathbf{a}$. Assume that $\mathbf{a}$ is not a path. Then, there exists a walk from u to v whose length is smaller than k.

Corollary 1.2.8 (When there is a walk, there is a path)**.** Let D be a multidigraph. Let u and v be two vertices of D. Assume that there is a walk from u to v of length k for some $k \in \mathbb{N}$. Then, there is a path from u to v of length $\leq k$.

The proofs of these facts are the same as for multigraphs.

The following proposition is an analogue of Proposition 1.2.4 from Lecture 4 for multidigraphs:

Proposition 1.2.9. Let D be a multidigraph. Let $\mathbf{w}$ be a walk of D. Then, $\mathbf{w}$ either is a path or contains a cycle (i.e., there exists a cycle of D whose arcs are arcs of $\mathbf{w}$).

Proof. This follows by the same argument as Proposition 1.2.4 in Lecture 4. □

1.3. Connectivity

We defined the "path-connected" relation for undirected graphs using the existence of paths. For a digraph, however, the relations "there is a walk from u to v" and "there is a walk from v to u" are (in general) distinct and non-symmetric, so I prefer not to give them a symmetric-looking symbol such as $\simeq_D$. Instead, we define **strong path-connectedness** to mean the existence of **both** walks:

Definition 1.3.1. Let D be a multidigraph. We define a binary relation $\simeq_D$ on the set $\operatorname{V}(D)$ as follows: For two vertices u and v of D, we shall have $u \simeq_D v$ if and only if there exists a walk from u to v in D and there exists a walk from v to u in D.

This binary relation $\simeq_D$ is called "**strong path-connectedness**". When two vertices u and v satisfy $u \simeq_D v$, we say that "u and v are **strongly path-connected**".

Example 1.3.2. Let D be as in Example 1.2.5. Then, $1 \simeq_D 2$, because there exists a walk from 1 to 2 in D (for instance, $(1, a, 2)$) and there also exists a walk from 2 to 1 in D (for instance, $(2, b, 3, d, 1)$). However, we don't have $3 \simeq_D 4$. Indeed, while there exists a walk from 3 to 4 in D, there exists no walk from 4 to 3 in D.

Proposition 1.3.3. Let D be a multidigraph. Then, the relation $\simeq_D$ is an equivalence relation.

Proof. Easy, like for simple graphs. □

Again, we can replace "walk" by "path" in the definition of the relation $\simeq_D$:

Proposition 1.3.4. Let D be a multidigraph. Let u and v be two vertices of D. Then, $u \simeq_D v$ if and only if there exist a path from u to v and a path from v to u.

Proof. Easy, like for simple graphs. □

Definition 1.3.5. Let D be a multidigraph. The equivalence classes of the equivalence relation $\simeq_D$ are called the **strong components** of D.

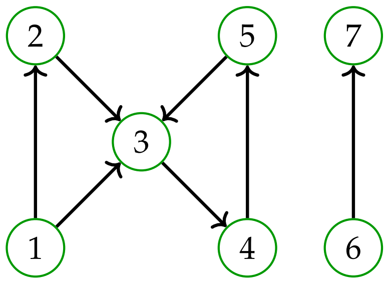
Definition 1.3.6. Let D be a multidigraph. We say that D is **strongly connected** if D has exactly one strong component.

Thus, a multidigraph D is strongly connected if and only if it has at least one vertex and there is a path from any vertex to any vertex.

In comparison, here is a weaker notion of connected components and connectivity:

Definition 1.3.7. Let D be a multidigraph. Consider its underlying undirected multigraph D^{und}. The components of this undirected multigraph D^{und} (that is, the equivalence classes of the equivalence relation $\simeq_{D^{\text{und}}}$) are called the **weak components** of D. We say that D is **weakly connected** if D has exactly one weak component (i.e., if D^{und} is connected).

Example 1.3.8. Let D be the following simple digraph:

$D =$  .

We treat D as a multidigraph (namely, D^{mult}).

The weak components of D are $\{1, 2, 3, 4, 5\}$ and $\{6, 7\}$.

The strong components of D are $\{1\}$, $\{2\}$, $\{3, 4, 5\}$, $\{6\}$ and $\{7\}$. (Indeed, for example, we have $1 \not\simeq_D 2 \not\simeq_D 3$ but $3 \simeq_D 4 \simeq_D 5$.)

So D is neither strongly nor weakly connected, but has more strong than weak components.

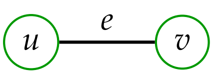
Example 1.3.9. The digraph from Example 1.2.2 is weakly connected, but not at all strongly connected (indeed, each of its strong components has size 1). The digraph from Example 1.2.3, on the other hand, is strongly connected.

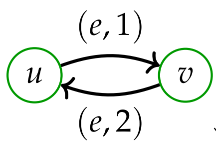
Proposition 1.3.10. Any strongly connected digraph is weakly connected.

Proof. Let D be a multidigraph. Then, any walk of D is (or, more precisely, gives rise to) a walk of D^{und}. Hence, if two vertices u and v of D are strongly path-connected in D, then they are path-connected in D^{und}. Therefore, if D is strongly connected, then D^{und} is connected, but this means that D is weakly connected. $\square$

Let us take a look at what bidirectionalization (i.e., the operation $G \mapsto G^{\text{bidir}}$ that sends a multigraph G to the multidigraph G^{bidir}) does to walks, paths, closed walks and cycles:

Proposition 1.3.11. Let G be a multigraph. Then:

- **(a)** The walks of G are "more or less the same as" the walks of the multidigraph G^{bidir}. More precisely, each walk of G gives rise to a walk of G^{bidir} (with the same starting point and the same ending point), and conversely, each walk of G^{bidir} gives rise to a walk of G. If G has no loops, then this is a one-to-one correspondence (i.e., a bijection) between the walks of G and the walks of G^{bidir}.
- **(b)** The paths of G are "more or less the same as" the paths of the multidigraph G^{bidir}. This is always a one-to-one correspondence, since paths cannot contain loops.
- **(c)** The closed walks of G are "more or less the same as" the closed walks of the multidigraph G^{bidir}.
- **(d)** The cycles of G are not quite the same as the cycles of G^{bidir}. In fact, if e is an edge of G with two distinct endpoints u and v, then (u, e, v, e, u) is not a cycle of G, but either $(u, (e, 1), v, (e, 2), u)$ or $(u, (e, 2), v, (e, 1), u)$ is a cycle of G^{bidir} (this is best seen on a picture: G has the edge 

whereas G^{bidir} has the arc-pair 

), so G^{bidir} usually has more cycles than G has. But it is true that each cycle of G gives rise to a cycle of G^{bidir}.

1.4. Eulerian walks and circuits

We have studied Eulerian walks and circuits for (undirected) multigraphs in Lectures 8 and 9. Let us now define analogous concepts for multidigraphs:

Definition 1.4.1. Let D be a multidigraph.

- **(a)** A walk of D is said to be **Eulerian** if each arc of D appears exactly once in this walk.

  (In other words: A walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ of D is said to be **Eulerian** if for each arc a of D, there exists exactly one $i \in \{1, 2, \ldots, k\}$ such that $a = a_i$.)

(b) An **Eulerian circuit** of D means a circuit (i.e., closed walk) of D that is Eulerian.

The Euler–Hierholzer theorem gives a necessary and sufficient criterion for a multigraph to have an Eulerian circuit or walk. For multidigraphs, there is an analogous result:

Theorem 1.4.2 (diEuler, diHierholzer)**.** Let D be a weakly connected multidigraph. Then:

(a) The multidigraph D has an Eulerian circuit if and only if each vertex v of D satisfies $\deg^+ v = \deg^- v$.

(b) The multidigraph D has an Eulerian walk if and only if all but two vertices v of D satisfy $\deg^+ v = \deg^- v$, and the remaining two vertices v satisfy $\left|\deg^+ v - \deg^- v\right| \leq 1$.

Proof. Homework set #4. □

Incidentally, the "each vertex v of D satisfies $\deg^+ v = \deg^- v$" condition has a name:

Definition 1.4.3. A multidigraph D is said to be **balanced** if each vertex v of D satisfies $\deg^+ v = \deg^- v$.

So balancedness is necessary and sufficient for the existence of an Eulerian circuit in a weakly connected multidigraph.

The following proposition is obvious:

Proposition 1.4.4. Let G be a multigraph. Then, the multidigraph G^{bidir} is balanced.

Proof. The definition of G^{bidir} yields that each vertex v of G^{bidir} satisfies $\deg^+ v = \deg v$ and $\deg^- v = \deg v$, where $\deg v$ denotes the degree of v as a vertex of G. Hence, each vertex v of G^{bidir} satisfies $\deg^+ v = \deg v = \deg^- v$. In other words, G^{bidir} is balanced. □

Combining this proposition with Theorem 1.4.2 **(a)**, we can obtain a curious fact about undirected(!) multigraphs:

Theorem 1.4.5. Let G be a connected multigraph. Then, the multidigraph G^{bidir} has an Eulerian circuit. In other words, there is a circuit of G that contains each edge **exactly twice**, and uses it once in each direction.

Proof. The multidigraph G^{bidir} is balanced (as we just saw) and weakly connected (this follows easily from the connectedness of G). Hence, Theorem 1.4.2 **(a)** can be applied to $D = G^{\text{bidir}}$. Thus, G^{bidir} has a Eulerian circuit. Reinterpreting this circuit as a circuit of G, we obtain a circuit of G that contains each edge **exactly twice**, and uses it once in each direction. This proves the theorem. □

1.5. Hamiltonian cycles and paths

We can define Hamiltonian paths and cycles for simple digraphs in the same way as we defined them for simple graphs:

Definition 1.5.1. Let $D = (V, A)$ be a simple digraph.

(a) A **Hamiltonian path** in D means a walk of D that contains each vertex of D exactly once. Obviously, it is a path.

(b) A **Hamiltonian cycle** in D means a cycle $(v_0, v_1, \ldots, v_k)$ of D such that each vertex of D appears exactly once among $v_0, v_1, \ldots, v_{k-1}$.

Convention 1.5.2. In the following, we will abbreviate:

- "Hamiltonian path" as "**hamp**";
- "Hamiltonian cycle" as "**hamc**".

We might wonder what can be said about hamps and hamcs for digraphs. Is there an analogue of Ore's theorem? The answer is "yes", but it is significantly harder to prove:

Theorem 1.5.3 (Meyniel)**.** Let $D = (V, A)$ be a strongly connected loopless simple digraph with n vertices. Assume that for each pair $(u, v) \in V \times V$ of two vertices u and v satisfying $u \neq v$ and $(u, v) \notin A$ and $(v, u) \notin A$, we have $\deg u + \deg v \geq 2n - 1$. Here, $\deg w$ means $\deg^+ w + \deg^- w$. Then, D has a hamc.

For the (rather complicated) proof of this, see [BonTho77] or [Berge91, §10.3, Theorem 7]. Note that the "strongly connected" condition is needed.

References

[Berge91] Claude Berge, *Graphs*, North-Holland Mathematical Library **6.1**, 3rd edition, North-Holland 1991.

[BonTho77] J. A. Bondy, C. Thomassen, *A short proof of Meyniel's theorem*, Discrete Mathematics **19**, Issue 2, 1977, pp. 195–197.