Math 530 Spring 2022, Lecture 1: Simple graphs

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

0.1. Plan

This is a course on **graphs** – a rather elementary concept (actually a cluster of closely related concepts) that can be seen all over mathematics. We will discuss several kinds of graphs (simple graphs, multigraphs, directed graphs, etc.) and study their features and properties. In particular, we will encounter walks on graphs, matchings of graphs, flows on networks (networks are graphs with extra data), and take a closer look at certain types of graphs such as trees and tournaments.

The theory of graphs goes back at least to Leonhard Euler, who in a 1736 paper [Euler36] (see [Euler53] for an English translation) solved a puzzle about an optimal tour of the town of Königsberg. It saw some more developments in the 19th century and straight-up exploded in the 20th; now it is one of the most active fields of mathematics.

We won't follow any book. But we will follow (to an extent) my lecture notes from Spring 2017: `https://www.cip.ifi.lmu.de/~grinberg/t/17s` (but keep in mind that they were written for a semester, not a quarter). The first few lectures will occasionally follow a stub of a text [17s] that I started writing back then. A long list of books appears on the course website; you don't strictly need any of them, but it's worth skimming them to get a feel for the topic (far beyond what we can do in this course) and learn more about directions you care about.

A few **administrativa:**

- See the website (`https://www.cip.ifi.lmu.de/~grinberg/t/22s`) for any info you might be looking for.

  We will use gradescope for HW. Blackboard will only be used for archiving announcement emails.

- Please interrupt whenever something is unclear!

  Don't hesitate to email questions either.

0.2. Notations

Notations:

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Thus, $0 \in \mathbb{N}$.

- The size (i.e., cardinality) of a finite set S is denoted by $|S|$.

- If S is a set, then the **powerset** of S means the set of all subsets of S. This powerset will be denoted by $\mathcal{P}(S)$.

  Moreover, if S is a set, and k is an integer, then $\mathcal{P}_k(S)$ will mean the set of all k-element subsets of S. For instance,

  $$\mathcal{P}_2(\{1,2,3\}) = \{\{1,2\}, \{1,3\}, \{2,3\}\}.$$

- For any number n and any $k \in \mathbb{N}$, we define the **binomial coefficient** $\binom{n}{k}$ to be the number

  $$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{\prod_{i=0}^{k-1}(n-i)}{k!}.$$

  These binomial coefficients have many interesting properties, which can often be found in textbooks on enumerative combinatorics. Some of the most important ones are the following:

  - The factorial formula: If $n, k \in \mathbb{N}$ and $n \geq k$, then $\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$.
  - The combinatorial interpretation: If $n, k \in \mathbb{N}$, and if S is an n-element set, then $\binom{n}{k}$ is the number of all k-element subsets of S (in other words, $|\mathcal{P}_k(S)| = \binom{n}{k}$).
  - Pascal's recursion: For any number n and any positive integer k, we have
    $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

1. Simple graphs

1.1. Definitions

The first type of graphs that we will consider are the "simple graphs", named so because of their very simple definition:

Definition 1.1.1. A **simple graph** is a pair (V, E), where V is a finite set, and where E is a subset of $\mathcal{P}_2(V)$.

To remind, $\mathcal{P}_2(V)$ is the set of all 2-element subsets of V. Thus, a simple graph is a pair (V, E), where V is a finite set, and E is a set consisting of 2-element subsets of V. We will abbreviate the word "simple graph" as "graph" in the next few lectures, but afterwards we will learn some more advanced and general notions of "graphs".

Example 1.1.2. Here is a simple graph:

$$(\{1,2,3,4\}, \ \{\{1,3\}, \ \{1,4\}, \ \{3,4\}\}).$$

Example 1.1.3. For any $n \in \mathbb{N}$, we can define a simple graph Cop_n to be the pair (V, E), where $V = \{1, 2, \ldots, n\}$ and

$$E = \{\{u, v\} \in \mathcal{P}_2(V) \ | \ \gcd(u, v) = 1\}.$$

We call this the **n-th coprimality graph**.

(Some authors do not require V to be finite in Definition 1.1.1; this leads to **infinite graphs**. But I shall leave this can of worms closed for this quarter.)

The purpose of simple graphs is to encode relations on a finite set – specifically the kind of relations that are binary (i.e., relate pairs of elements), symmetric (i.e., mutual) and irreflexive (i.e., an element cannot be related to itself). For example, the graph Cop_n in Example 1.1.3 encodes the coprimality (aka coprimeness) relation on the set $\{1, 2, \ldots, n\}$, except that the latter relation is not irreflexive (1 is coprime to 1, but $\{1, 1\}$ is not in E; thus, the graph Cop_n "forgets" that 1 is coprime to 1). For another example, if V is a set of people, and E is the set of $\{u, v\} \in \mathcal{P}_2(V)$ such that u has been married to v at some point, then (V, E) is a simple graph. Even in 2022, marriage to oneself is not a thing, so all marriages can be encoded as 2-element subsets.[1]

The following notations provide a quick way to reference the elements of V and E when given a graph (V, E):

Definition 1.1.4. Let $G = (V, E)$ be a simple graph.

(a) The set V is called the **vertex set** of G; it is denoted by $\operatorname{V}(G)$. (Notice that the letter "V" in "$\operatorname{V}(G)$" is upright, as opposed to the letter "V" in "(V, E)", which is italic. These are two different symbols, and have different meanings: The letter V stands for the specific set V which is the first component of the pair G, whereas the letter V is part of the notation $\operatorname{V}(G)$ for the vertex set of any graph. Thus, if $H = (W, F)$ is another graph, then $\operatorname{V}(H)$ is W, not V.)

The elements of V are called the **vertices** (or the **nodes**) of G.

(b) The set E is called the **edge set** of G; it is denoted by $\operatorname{E}(G)$. (Again, the letter "E" in "$\operatorname{E}(G)$" is upright, and stands for a different thing than the "E".)

[1]The more standard example for a social graph would be a "friendship graph"; here, V is again a set of people, but E is now the set of $\{u, v\} \in \mathcal{P}_2(V)$ such that u and v are friends. Of course, this only works if you think of friendship as being automatically mutual (true for facebook friendship, questionable for the actual thing).

The elements of E are called the **edges** of G. When u and v are two elements of V, we shall often use the notation uv for $\{u, v\}$; thus, each edge of G has the form uv for two distinct elements u and v of V. Of course, we always have $uv = vu$.

Notice that each simple graph G satisfies $G = (\operatorname{V}(G), \operatorname{E}(G))$.

(c) Two vertices u and v of G are said to be **adjacent** (to each other) if $uv \in E$ (that is, if uv is an edge of G). In this case, the edge uv is said to **join** u with v (or **connect** u and v); the vertices u and v are called the **endpoints** of this edge. When the graph G is not obvious from the context, we shall often say "adjacent in G" instead of just "adjacent".

Two vertices u and v of G are said to be **non-adjacent** (to each other) if they are not adjacent (i.e., if $uv \notin E$).

(d) Let v be a vertex of G (that is, $v \in V$). Then, the **neighbors** of v (in G) are the vertices u of G that satisfy $vu \in E$. In other words, the **neighbors** of v are the vertices of G that are adjacent to v.

Example 1.1.5. Let G be the simple graph

$$(\{1,2,3,4\}, \ \{\{1,3\}, \ \{1,4\}, \ \{3,4\}\})$$

from Example 1.1.2. Then, its vertex set and its edge set are

$$\operatorname{V}(G) = \{1,2,3,4\} \quad \text{and} \quad \operatorname{E}(G) = \{\{1,3\}, \ \{1,4\}, \ \{3,4\}\} = \{13, \ 14, \ 34\}$$

(using our notation uv for $\{u, v\}$). The vertices 1 and 3 are adjacent (since $13 \in \operatorname{E}(G)$), but the vertices 1 and 2 are not (since $12 \notin \operatorname{E}(G)$). The neighbors of 1 are 3 and 4. The endpoints of the edge 34 are 3 and 4.

1.2. Drawing graphs

There is a common method to represent graphs visually: Namely, a graph can be drawn as a set of points in the plane and a set of curves connecting some of these points with each other.

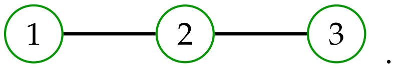
More precisely:

Definition 1.2.1. A simple graph G can be visually represented by **drawing** it on the plane. To do so, we represent each vertex of G by a point (at which we put the name of the vertex), and then, for each edge uv of G, we draw a curve that connects the point representing u with the point representing v. The positions of the points and the shapes of the curves can be chosen freely, as long as they allow the reader to unambiguously reconstruct the graph G

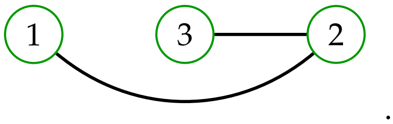
from the picture. (Thus, for example, the curves should not pass through any points other than the ones they mean to connect.)

Example 1.2.2. Let us draw some simple graphs.

(a) The simple graph $(\{1,2,3\}, \{12,23\})$ (where we are again using the shorthand notation uv for $\{u,v\}$) can be drawn as follows:

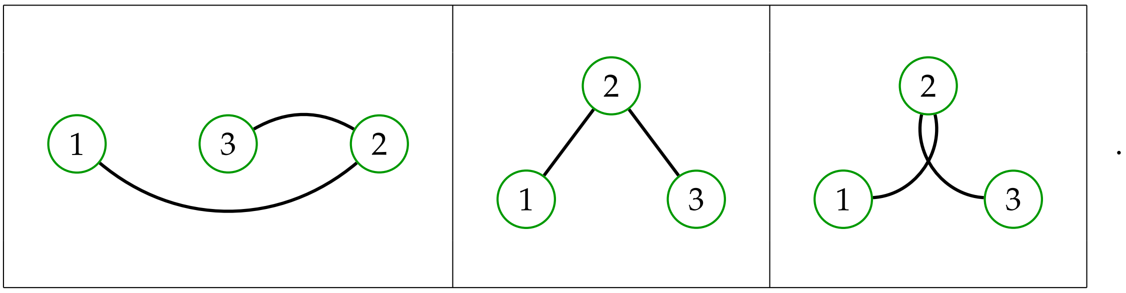


This is (in a sense) the simplest way to draw this graph: The edges are represented by straight lines. But we can draw it in several other ways as well – e.g., as follows:

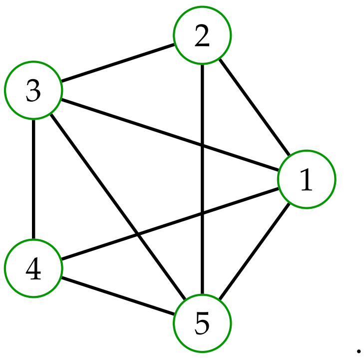


Here, we have placed the points representing the vertices $1,2,3$ differently. As a consequence, we were not able to draw the edge 12 as a straight line, because it would then have overlapped with the vertex 3, which would make the graph ambiguous (the edge 12 could be mistaken for two edges 13 and 32).

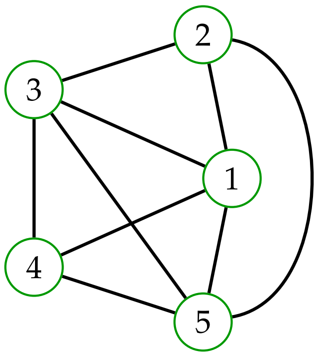
Here are three further drawings of the same graph $(\{1,2,3\}, \{12,23\})$:



(b) Consider the 5-th coprimality graph Cop_5 defined in Example 1.1.3. Here is one way to draw it:



Here is another way to draw the same graph Cop_5, with fewer intersections between edges:



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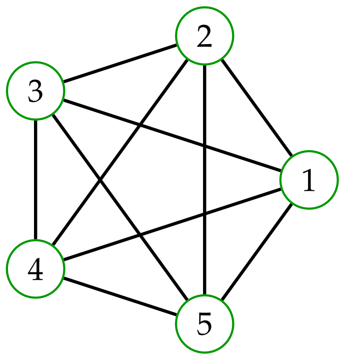
By appropriately repositioning the points corresponding to the five vertices of Cop_5, we can actually get rid of all intersections and make all the edges straight (as opposed to curved). Can you find out how?

(c) Let us draw one further graph: the simple graph $(\{1,2,3,4,5\}, \mathcal{P}_2(\{1,2,3,4,5\}))$. This is the simple graph whose vertices are $1,2,3,4,5$, and whose edges are all possible two-element sets consisting of its vertices (i.e., each pair of two distinct vertices is adjacent). We shall later call this graph the "complete graph K_5". Here is a simple way to draw this graph:



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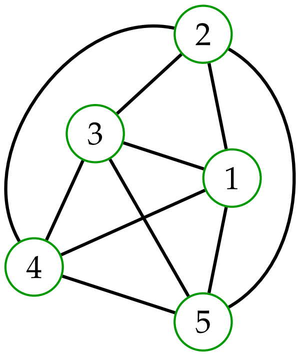
This drawing is useful for many purposes; for example, it makes the abstract symmetry of this graph (i.e., the fact that, roughly speaking, its vertices $1,2,3,4,5$ are "equal in rights") obvious. But sometimes, you might want to draw it differently, to minimize the number of intersecting curves. Here is a drawing with fewer intersections:



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In this drawing, we have only one intersection between two curves left. Can we get rid of all intersections?

This is a question of topology, not of combinatorics, since it really is about curves in the plane rather than about finite sets and graphs. The answer is "no". (That is, no matter how you draw this graph in the plane, you will always have at least one pair of curves intersect.) This is a classical result (one of the first theorems in the theory of **planar graphs**), and proofs of it can be found in various textbooks (e.g., [FriFri98, Theorem 4.1.2], which is generally a good introduction to planar graph theory even if it uses terminology somewhat different from ours). Note that any proof must use some analysis or topology, since the result relies on the notion of a (continuous) curve in the plane (if curves were allowed to be non-continuous, then they could "jump over" one another, so they could easily avoid intersecting!).

1.3. A first fact: The Ramsey number $R(3,3) = 6$

Enough definitions; let's state a first result:

Proposition 1.3.1. Let G be a simple graph with $|\operatorname{V}(G)| \geq 6$ (that is, G has at least 6 vertices). Then, at least one of the following two statements holds:

- *Statement 1:* There exist three distinct vertices a, b and c of G such that ab, bc and ca are edges of G.
- *Statement 2:* There exist three distinct vertices a, b and c of G such that none of ab, bc and ca is an edge of G.

In other words, Proposition 1.3.1 says that if a graph G has at least 6 vertices, then we can either find three distinct vertices that are mutually adjacent[2] or find

[2]by which we mean (of course) that any two **distinct** ones among these three vertices are adjacent

three distinct vertices that are mutually non-adjacent (i.e., no two of them are adjacent), or both. Often, this is restated as follows: "In any group of at least six people, you can always find three that are (pairwise) friends to each other, or three no two of whom are friends" (provided that friendship is a symmetric relation).

We will give some examples in a moment, but first let us introduce some convenient terminology:

Definition 1.3.2. Let G be a simple graph.

(a) A set $\{a, b, c\}$ of three distinct vertices of G is said to be a **triangle** (of G) if every two distinct vertices in this set are adjacent (i.e., if ab, bc and ca are edges of G).

(b) A set $\{a, b, c\}$ of three distinct vertices of G is said to be an **anti-triangle** (of G) if no two distinct vertices in this set are adjacent (i.e., if none of ab, bc and ca is an edge of G).

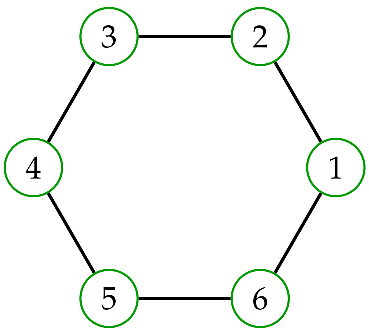
Thus, Proposition 1.3.1 says that every simple graph with at least 6 vertices contains a triangle or an anti-triangle (or both).

Example 1.3.3. Let us show two examples of graphs G to which Proposition 1.3.1 applies, as well as an example to which it does not:

(a) Let G be the graph (V, E), where

$$V = \{1, 2, 3, 4, 5, 6\} \qquad \text{and}$$
$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}\}.$$

(This graph can be drawn in such a way as to look like a hexagon:



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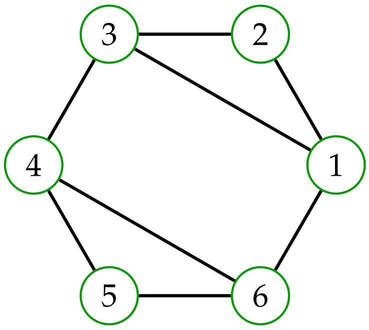
) This graph satisfies Proposition 1.3.1, since $\{1, 3, 5\}$ is an anti-triangle (or since $\{2, 4, 6\}$ is an anti-triangle).

(b) Let G be the graph (V, E), where

$$V = \{1,2,3,4,5,6\} \qquad \text{and}$$
$$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,1\}, \{1,3\}, \{4,6\}\}.$$

(This graph can be drawn in such a way as to look like a hexagon with two extra diagonals:

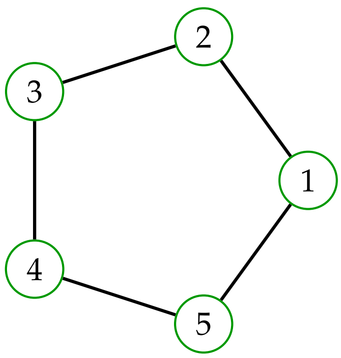


) This graph satisfies Proposition 1.3.1, since $\{1,2,3\}$ is a triangle.

(c) Let G be the graph (V, E), where

$$V = \{1,2,3,4,5\} \qquad \text{and}$$
$$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}.$$

(This graph can be drawn to look like a pentagon:



) Proposition 1.3.1 says nothing about this graph, since this graph does not satisfy the assumption of Proposition 1.3.1 (in fact, its number of vertices $|\mathrm{V}(G)|$ fails to be ≥ 6). By itself, this does not yield that the claim of Proposition 1.3.1 is false for this graph. However, it is easy to check that the claim actually **is** false for this graph: It has neither a triangle nor an anti-triangle.

Proof of Proposition 1.3.1. We need to prove that G has a triangle or an anti-triangle (or both).

Choose any vertex $u \in \operatorname{V}(G)$. (This is clearly possible, since $|\operatorname{V}(G)| \geq 6 \geq 1$.) Then, there are at least 5 vertices distinct from u (since G has at least 6 vertices). We are in one of the following two cases:

Case 1: The vertex u has at least 3 neighbors.

Case 2: The vertex u has at most 2 neighbors.

Let us consider Case 1 first. In this case, the vertex u has at least 3 neighbors. Hence, we can find three distinct neighbors p, q and r of u. Consider these p, q and r. If one (or more) of pq, qr and rp is an edge of G, then G has a triangle (for example, if pq is an edge of G, then $\{u, p, q\}$ is a triangle). If not, then G has an anti-triangle (namely, $\{p, q, r\}$). Thus, in either case, our proof is complete in Case 1.

Let us now consider Case 2. In this case, the vertex u has at most 2 neighbors. Hence, the vertex u has at least 3 non-neighbors[3] (since there are at least 5 vertices distinct from u in total). Thus, we can find three distinct non-neighbors p, q and r of u. Consider these p, q and r. If all of pq, qr and rp are edges of G, then G has a triangle (namely, $\{p, q, r\}$). If not, then G has an anti-triangle (for example, if pq is not an edge of G, then $\{u, p, q\}$ is an anti-triangle). In either case, we are thus done with the proof in Case 2. Thus, both cases are resolved, and the proof is complete. □

Notice the symmetry between Case 1 and Case 2 in our above proof: the arguments used were almost the same, except that neighbors and non-neighbors swapped roles.

Remark 1.3.4. Proposition 1.3.1 could also be proved by brute force as well (using a computer). Indeed, it clearly suffices to prove it for all simple graphs with 6 vertices (as opposed to ≥ 6 vertices), because if a graph has more than 6 vertices, then we can just throw away some of them until we have only 6 left. However, there are only finitely many simple graphs with 6 vertices (up to relabeling of their vertices), and the validity of Proposition 1.3.1 can be checked for each of them. This is, of course, cumbersome (even a computer would take a moment checking all the 2^{15} possible graphs for triangles and anti-triangles) and unenlightening.

Proposition 1.3.1 is the first result in a field of graph theory known as *Ramsey theory*. I shall not dwell on this field in this course, but let me make a few more remarks. The first step beyond Proposition 1.3.1 is the following generalization:

Proposition 1.3.5. Let r and s be two positive integers. Let G be a simple graph with $|\operatorname{V}(G)| \geq \dbinom{r+s-2}{r-1}$. Then, at least one of the following two statements holds:

[3]The word "non-neighbor" shall here mean a vertex that is not adjacent to u and **distinct from** u. Thus, u does not count as a non-neighbor of u.

- *Statement 1:* There exist r distinct vertices of G that are mutually adjacent (i.e., each two distinct ones among these r vertices are adjacent).
- *Statement 2:* There exist s distinct vertices of G that are mutually non-adjacent (i.e., no two distinct ones among these s vertices are adjacent).

Applying Proposition 1.3.5 to $r = 3$ and $s = 3$, we can recover Proposition 1.3.1.

One might wonder whether the number $\dbinom{r+s-2}{r-1}$ in Proposition 1.3.5 can be improved – i.e., whether we can replace it by a smaller number without making Proposition 1.3.5 false. In the case of $r = 3$ and $s = 3$, this is impossible, because the number 6 in Proposition 1.3.1 cannot be made smaller[4]. However, for some other values of r and s, the value $\dbinom{r+s-2}{r-1}$ can be improved. (For example, for $r = 4$ and $s = 4$, the best possible value is 18 rather than $\dbinom{4+4-2}{4-1} = 20$.) The smallest possible value that could stand in place of $\dbinom{r+s-2}{r-1}$ in Proposition 1.3.5 is called the **Ramsey number** $R(r,s)$; thus, we have just showed that $R(3,3) = 6$. Finding $R(r,s)$ for higher values of r and s is a hard computational challenge; here are some values that have been found with the help of computers:

$$R(3,4) = 9; \qquad R(3,5) = 14; \qquad R(3,6) = 18; \qquad R(3,7) = 23;$$
$$R(3,8) = 28; \qquad R(3,9) = 36; \qquad R(4,4) = 18; \qquad R(4,5) = 25.$$

(We are only considering the cases $r \leq s$, since it is easy to see that $R(r,s) = R(s,r)$ for all r and s. Also, the trivial values $R(1,s) = 1$ and $R(2,s) = s+1$ for $s \geq 2$ are omitted.) The Ramsey number $R(5,5)$ is still unknown (although it is known that $43 \leq R(5,5) \leq 48$).

Proposition 1.3.5 can be further generalized to a result called *Ramsey's theorem*. The idea behind the generalization is to slightly change the point of view, and replace the simple graph G by a complete graph (i.e., a simple graph in which every two distinct vertices are adjacent) whose edges are colored in two colors (say, blue and red). This is a completely equivalent concept, because the concepts of "adjacent" and "non-adjacent" in G can be identified with the concepts of "adjacent through a blue edge" (i.e., the edge connecting them is colored blue) and "adjacent through a red edge", respectively. Statements 1 and 2 then turn into "there exist r distinct vertices that are mutually adjacent through blue edges" and "there exist s distinct vertices that are mutually adjacent through red edges", respectively. From this point of view,

[4]Indeed, we saw in Example 1.3.3 **(c)** that 5 vertices would not suffice.

it is only logical to generalize Proposition 1.3.5 further to the case when the edges of a complete graph are colored in k (rather than two) colors. The corresponding generalization is known as Ramsey's theorem. We refer to the well-written Wikipedia page `https://en.wikipedia.org/wiki/Ramsey's_theorem` for a treatment of this generalization with proof, as well as a table of known Ramsey numbers $R(r,s)$ and a self-contained (if somewhat terse) proof of Proposition 1.3.5. Ramsey's theorem can be generalized and varied further; this usually goes under the name "Ramsey theory". For elementary introductions, see the Cut-the-knot page `http://www.cut-the-knot.org/Curriculum/Combinatorics/ThreeOrThree.shtml` , the above-mentioned Wikipedia article, as well as the texts by Harju [Harju14], Bollobas [Bollob98] and West [West01].

There is one more direction in which Proposition 1.3.1 can be improved a bit: A graph G with at least 6 vertices has not only one triangle or anti-triangle, but at least two of them (this can include having one triangle and one anti-triangle). I posed this as a homework exercise (homework set #1, Exercise 1 **(a)**) in my Spring 2017 course; see the course page for solutions.

References

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[Bollob98] Béla Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics **184**, Springer 1998.

[Euler36] Leonhard Euler, *Solutio problematis ad geometriam situs pertinentis*, Euler Archive - All Works **53**, 1741.

[Euler53] Leonhard Euler, James R. Newman, *Leonhard Euler and the Koenigsberg bridges (English translation of "Solutio problematis ad geometriam situs pertinentis")*, Scientific American 1953.

[FriFri98] Rudolf Fritsch, Gerda Fritsch, *The Four-Color Theorem*, translated by Julie Peschke, Springer 1998.

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