Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2022: Homework 8 due 2022-06-08 at 11:00 AM

Please solve 4 of the 8 problems!

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1 Exercise 1

1.1 Problem

Let $D = (V, A, \psi)$ be a strongly connected multidigraph.

A *wealth distribution* on D shall mean a family $(k_v)_{v\in V}$ of integers (one for each vertex $v \in V$). If $k = (k_v)_{v\in V}$ is a wealth distribution, then we refer to each value k_v as the *wealth* of the vertex v, and we define the *total wealth* of k to be the sum $\sum_{v\in V} k_v$. We say that a vertex v is *in debt* in a given wealth distribution $k = (k_v)_{v\in V}$ if its wealth k_v is negative.

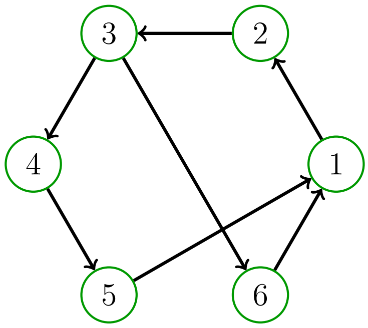
For any vertices v and w, we let $a_{v,w}$ denote the number of arcs that have source v and w.

A *donation* is an operation that transforms a wealth distribution as follows: We choose a vertex v, and we decrease its wealth by its outdegree $\deg^+ v$, and then increase the wealth of each vertex $w \in V$ (including v itself) by $a_{v,w}$. (You can think of v as donating a unit of wealth for each arc that has source v. This unit flows to the target to this arc. Note that a donation does not change the total wealth.)

Let k be a wealth distribution on D whose total wealth is larger than $|A| - |V|$. Prove that by an appropriately chosen finite sequence of donations, we can ensure that no vertex is in debt.

1.2 Remark

For instance, consider the digraph



with wealth distribution $(k_1, k_2, k_3, k_4, k_5, k_6) = (-1, -1, 1, 2, 0, 1)$. The vertices 1 and 2 are in debt here, but it is possible to get all vertices out of debt by having the vertices $4, 5, 6, 1$ donate in some order (the order clearly does not matter for the result[1]).

Note that vertices are allowed to donate multiple times (although in the above example, this was unnecessary).

1.3 Hint

Show first that if the total wealth is larger than $|A| - |V|$, then at least one vertex v has wealth $\geq \deg^+ v$.

1.4 Solution

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2 Exercise 2

2.1 Problem

We continue with the setting and terminology of Exercise 1.

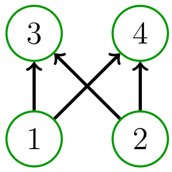
A *clawback* is an operation that transforms a wealth distribution as follows: We choose a vertex v, and we increase its wealth by its outdegree $\deg^+ v$, and then decrease the wealth of each vertex $w \in V$ (including v itself) by $a_{v,w}$. (Thus, a clawback is the inverse of a donation.)

Let k be a wealth distribution on D whose total wealth is larger than $|A| - |V|$. Prove that by an appropriately chosen finite sequence of clawbacks, we can ensure that no vertex is in debt.

[1] Depending on the order, some vertices will go into debt in the process, but this is okay as long as they ultimately end up debt-free.

2.2 Remark

Note that we are still assuming D to be strongly connected. Otherwise, the truth of the claim is not guaranteed. For instance, for the digraph



with wealth distribution $(k_1, k_2, k_3, k_4) = (0, 0, -1, 2)$, no sequence of donations and clawbacks will result in every vertex being out of debt (since the wealth difference $k_4 - k_3$ is preserved under any donation or clawback, but this difference is too large to come from a debt-free distribution with total weight 1).

2.3 Hint

Show that any donation is equivalent to an appropriately chosen composition of clawbacks.

2.4 Solution

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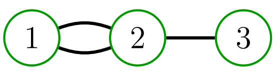
3 Exercise 3

3.1 Problem

Let $G = (V, E)$ be a simple graph such that each vertex of G has degree ≥ 1. Prove that there exists a subset S of V having size $\geq \sum_{v \in V} \dfrac{2}{1 + \deg v}$ and with the property that the induced subgraph $G[S]$ is a forest.

3.2 Remark

As the example of 

shows, this claim is not true for loopless multigraphs (unlike the similar Theorem 1.1.3 in Lecture 23).

3.3 Solution

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4 Exercise 4

4.1 Problem

Let n be a positive integer. Prove that there exists a tournament with n vertices and at least $\dfrac{n!}{2^{n-1}}$ Hamiltonian paths.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let $G = (V, E, \varphi)$ be a multigraph. Let M be a matching of G.

An *augmenting path* for M shall mean a path $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ of G such that k is odd[2] and such that

- the even-indexed edges $e_2, e_4, \ldots, e_{k-1}$ belong to M (note that this condition is vacuously true if $k = 1$);
- the odd-indexed edges $e_1, e_3, \ldots, e_k$ belong to $E \setminus M$;
- neither the starting point v_0 nor the ending point v_k is matched in M.

Prove that M has maximum size among all matchings of G if and only if there exists no augmenting path for M.

5.2 Hint

If M and M' are two matchings of G, what can you say about the symmetric difference $(M \cup M') \setminus (M \cap M')$?

5.3 Solution

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[2] Note that $k = 1$ is allowed.

6 Exercise 6

6.1 Problem

Let (G, X, Y) be a bipartite graph. Prove that

$$\sum_{A \subseteq X} (-1)^{|A|} [N(A) = Y] = \sum_{B \subseteq Y} (-1)^{|B|} [N(B) = X]$$

(where we are using the Iverson bracket notation).

6.2 Solution

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7 Exercise 7

7.1 Problem

Let A and B be two finite sets such that $|B| \geq |A|$. Let $d_{i,j}$ be a real number for each $(i, j) \in A \times B$. Let[3]

$$m_1 = \min_{\sigma: A \to B \text{ injective}} \max_{i \in A} d_{i,\sigma(i)}$$

and

$$m_2 = \max_{\substack{I \subseteq A;\ J \subseteq B; \\ |I|+|J|=|B|+1}} \min_{(i,j) \in I \times J} d_{i,j}.$$

Prove that $m_1 = m_2$.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let p be a prime number. Let $(a_1, a_2, a_3, \ldots)$ be a sequence of integers that is periodic with period p (that is, that satisfies $a_i = a_{i+p}$ for each $i > 0$). Assume that $a_1 + a_2 + \cdots + a_p$ is not divisible by p. Prove that there exists an $i \in \{1, 2, \ldots, p\}$ such that none of the p numbers

$$a_i,\ a_i + a_{i+1},\ a_i + a_{i+1} + a_{i+2},\ \ldots,\ a_i + a_{i+1} + \cdots + a_{i+p-1}$$

(that is, of the p sums $a_i + a_{i+1} + \cdots + a_j$ for $i \leq j < i + p$) is divisible by p.

[3] The notation "$\min_{\text{some kind of objects}}$ some kind of value" means the minimum of the given value over all objects of the given kind. An analogous notation is used for a maximum.

8.2 Remark

This would be false if p was not prime. For instance, for $p = 4$, the sequence $(0, 2, 2, 2, 0, 2, 2, 2, \ldots)$ would be a counterexample.

8.3 Hint

What is the digraph, and why does it have a cycle?

8.4 Solution

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