Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2022: Homework 6
due 2022-05-13 at 11:00 AM
Please solve 3 of the 6 problems!

Darij Grinberg

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1 Exercise 1

1.1 Problem

Let G be a connected multigraph with an even number of vertices. Prove that there exists a spanning subgraph H of G such that each vertex of H has odd degree (in H).

1.2 Hint

The problem can be reduced to the case when G is a tree (how?).

1.3 Solution

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2 Exercise 2

2.1 Problem

Let T be a tree. Let $\mathbf{p} = (p_0, *, p_1, *, p_2, \ldots, *, p_m)$ be a longest path of T. (We write asterisks for the edges since we don't need to name them.)

Prove the following:

(a) If m is even, then the only center of T is $p_{m/2}$.

(b) If m is odd, then the two centers of T are $p_{(m-1)/2}$ and $p_{(m+1)/2}$.

2.2 Remark

This result (due to Arthur Cayley in 1875) shows once again that each tree has exactly one center or two adjacent centers, and also shows that any two longest paths of a tree have a common vertex.

2.3 Solution

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3 Exercise 3

3.1 Problem

Let T be a tree. For any vertex v of T, we let c_v denote the size of the largest component[1] of the graph $T \setminus v$. (Recall that $T \setminus v$ is the graph obtained from T by removing the vertex v and all edges that contain v.)

The vertices v of T that minimize the number c_v are called the *centroids* of T.

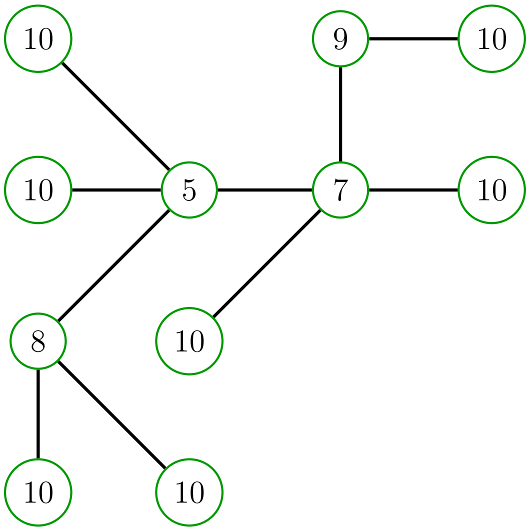
(a) Prove that T has no more than two centroids, and furthermore, if T has two centroids, then these two centroids are adjacent.

(b) Find a tree T such that the centroid(s) of T are distinct from the center(s) of T.

3.2 Remark

Here is an example of a tree T, where each vertex v is labelled with the corresponding number c_v:

[1]Note that a component (according to our definition) is a set of vertices; thus, its size is the number of vertices in it.



3.3 Hint

Part **(a)** is, of course, an analogue of the corresponding property of centers that we proved in Lecture 15. However, I don't think it can be solved in the same way. (Removing all leaves of a tree can affect the numbers c_v in rather unpredictable ways.) Instead, try to show that any two centroids of T must be adjacent.

3.4 Solution

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4 Exercise 4

4.1 Problem

Let $D = (V, A, \phi)$ be a multidigraph that has no cycles[2]. Let $r \in V$ be some vertex of D. Prove the following:

(a) If $\deg^- u > 0$ holds for all $u \in V \setminus \{r\}$, then r is a from-root of D.

(b) If $\deg^- u = 1$ holds for all $u \in V \setminus \{r\}$, then D is an arborescence rooted from r.

4.2 Solution

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[2] Recall that cycles in a digraph have to be directed cycles – i.e., each arc is traversed from its source to its target.

5 Exercise 5

5.1 Problem

Let $D = (V, A)$ be a simple digraph that has no cycles.

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a list of vertices of D (not necessarily a walk!), then a *back-cut* of $\mathbf{v}$ shall mean an arc $a \in A$ whose source is v_i and whose target is v_j for some $i, j \in \{1, 2, \ldots, n\}$ satisfying $i > j$. (Colloquially speaking, a back-cut of $\mathbf{v}$ is an arc of D that leads from some vertex of $\mathbf{v}$ to some earlier vertex of $\mathbf{v}$.)

A list $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ of vertices of D is said to be a *toposort*[3] of D if it contains each vertex of D exactly once and has no back-cuts.

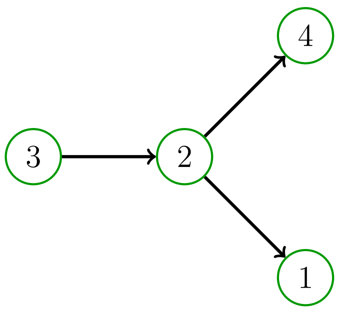
Prove the following:

(a) The digraph D has at least one toposort.

(b) If D has only one toposort, then this toposort is a Hamiltonian path of D.

5.2 Remark

For example, the digraph



has two toposorts: $(3, 2, 1, 4)$ and $(3, 2, 4, 1)$.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let n be a positive integer. Let $K_{n,2}$ be the simple graph with vertex set $\{1, 2, \ldots, n\} \cup \{-1, -2\}$ such that two vertices of $K_{n,2}$ are adjacent if and only if they have opposite signs (i.e., each positive vertex is adjacent to each negative vertex, but no two vertices of the same sign are adjacent). We regard $K_{n,2}$ as a multigraph in the usual way.

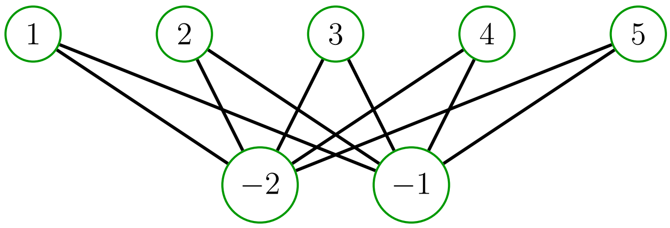
(a) Without using the matrix-tree theorem, prove that the number of spanning trees of $K_{n,2}$ is $n \cdot 2^{n-1}$.

(b) Let $K'_{n,2}$ be the graph obtained by adding a new edge $\{-1, -2\}$ to $K_{n,2}$. How many spanning trees does $K'_{n,2}$ have?

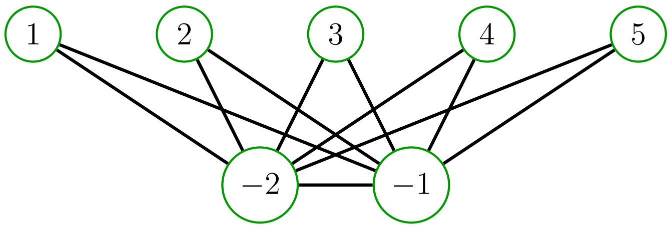
[3] This is short for "topological sorting". I don't know where this name comes from.

6.2 Remark

Here is the graph $K_{n,2}$ for $n = 5$:



And here is the corresponding graph $K'_{n,2}$:



6.3 Solution

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