Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2022: Homework 5 due 2022-05-04 at 11:00 AM **Please solve 3 of the 6 problems!**

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1 Exercise 1

1.1 Problem

Let $D = (V, A, \psi)$ be a multidigraph.

For two vertices u and v of D, we shall write $u \overset{*}{\to} v$ if there exists a path from u to v.

A *root* of D means a vertex $u \in V$ such that each vertex $v \in V$ satisfies $u \overset{*}{\to} v$.

A *common ancestor* of two vertices u and v means a vertex $w \in V$ such that $w \overset{*}{\to} u$ and $w \overset{*}{\to} v$.

Assume that D has at least one vertex. Prove that D has a root if and only if every two vertices in D have a common ancestor.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let G be a multigraph that has no loops. Assume that there exists a vertex u of G such that

for each vertex v of G, there is a **unique** path from u to v in G.

Prove that G is a tree.

2.2 Remark

Notice the quantifiers used here: $\exists u \forall v$. This differs from the $\forall u \forall v$ in Statement T2 of the tree equivalence theorem (Theorem 1.2.4 in Lecture 13).

2.3 Solution

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3 Exercise 3

3.1 Problem

Let F be any field[1].

Let $G = (V, E, \varphi)$ be a multigraph, where $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

For each edge $e \in E$, we construct a column vector $\chi_e \in F^n$ (that is, a column vector with n entries) as follows:

- If e is a loop, then we let χ_e be the zero vector.
- Otherwise, we let u and v be the two endpoints of e, and we let χ_e be the column vector that has a 1 in its u-th position, a -1 in its v-th position, and 0s in all other positions. (This depends on which endpoint we call u and which endpoint we call v, but we just make some choice and stick with it. The result will be true no matter how we choose.)

Let M be the $n \times |E|$-matrix over F whose columns are the column vectors χ_e for all $e \in E$ (we order them in some way; the exact order doesn't matter). Prove that

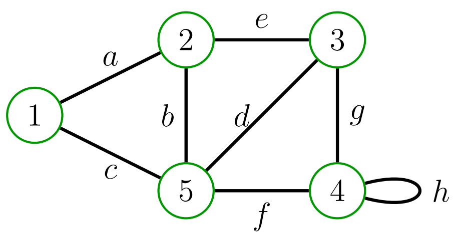
$$\operatorname{rank} M = |V| - \operatorname{conn} G.$$

(Recall that $\operatorname{conn} G$ denotes the number of components of G.)

[1] If you find it more convenient, you can assume that $F = \mathbb{R}$ or $F = \mathbb{C}$.

3.2 Remark

Here is an example: Let G be the multigraph



(so that $n = 5$). Then, if we choose the endpoints of b to be 2 and 5 in this order, then we have $\chi_b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$. (Choosing them to be 5 and 2 instead, we would obtain $\chi_b = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.)
If we do the same for all edges of G (that is, we choose the smaller endpoint as u and the larger endpoint as v), and if we order the columns so that they correspond to the edges a, b, c, d, e, f, g, h from left to right, then the matrix M comes out as follows:

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $\operatorname{rank} M = 4$, which is precisely $|V| - \operatorname{conn} G$.

Another way of putting the claim of the exercise is that the span of the vectors χ_e for all $e \in E$ has dimension $|V| - \operatorname{conn} G$.

Topologists will recognize the matrix M as (a matrix that represents) the boundary operator $\partial : C_1(G) \to C_0(G)$, where G is viewed as a CW-complex.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let $G = (V, E, \varphi)$ be a connected multigraph such that $|E| \geq |V|$. Show that there exists an injective map $f : V \to E$ such that for each vertex $v \in V$, the edge $f(v)$ contains v.

(In other words, show that we can assign to each vertex an edge that contains this vertex in such a way that no edge is assigned twice.)

4.2 Solution

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5 Exercise 5

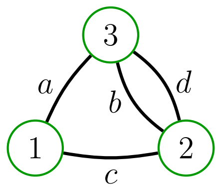
5.1 Problem

Let G be a connected multigraph. Let $\mathcal{S}$ be the simple graph whose vertices are the spanning trees of G, and whose edges are defined as follows: Two spanning trees T_1 and T_2 of G are adjacent (as vertices of $\mathcal{S}$) if and only if T_2 can be obtained from T_1 by removing an edge and adding another (i.e., if and only if there exist an edge e_1 of T_1 and an edge e_2 of T_2 such that $e_2 \neq e_1$ and $T_2 \setminus e_2 = T_1 \setminus e_1$).

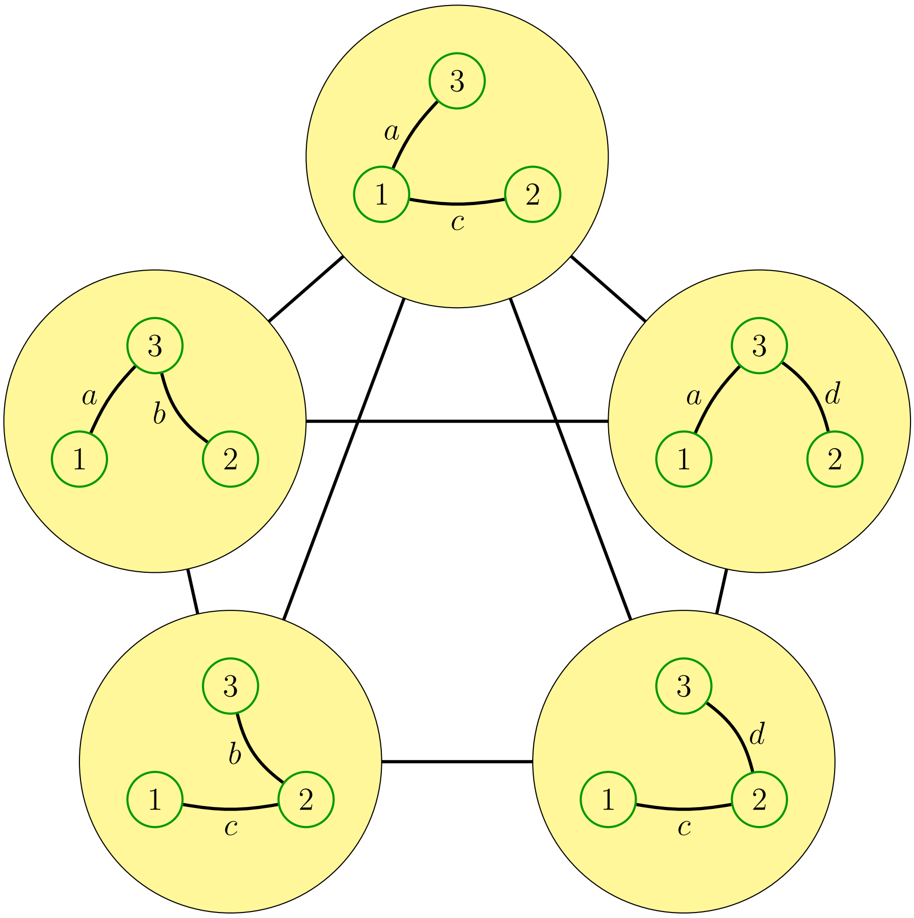
Prove that the simple graph $\mathcal{S}$ is itself connected. (In simpler language: Prove that any spanning tree of G can be transformed into any other spanning tree of G by a sequence of legal "remove an edge and add another" operations, where such an operation is called *legal* if its result is a spanning tree of G.)

5.2 Remark

Let G be the following multigraph:



Then, the graph $\mathcal{S}$ looks as follows:



5.3 Hint

You must show that any two spanning trees T_1 and T_2 are path-connected in $\mathcal{S}$. Induct on the number of edges of T_1 that are not edges of T_2.

5.4 Solution

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6 Exercise 6

6.1 Problem

Let $G = (V, E, \varphi)$ be a connected multigraph. Let $w : E \to \mathbb{R}$ be a map that assigns a real number $w(e)$ to each edge e. We shall call this real number $w(e)$ the *weight* of the edge e.

If $H = (W, F, \varphi \mid_F)$ is a subgraph of G, then the *weight* $w(H)$ of H is defined to be $\sum\limits_{f \in F} w(f)$ (that is, the sum of the weights of all edges of H).

A *w-minimum spanning tree* of G means a spanning tree of G that has the smallest weight among all spanning trees of G.

In Lecture 14, we have seen a way to construct a spanning tree of G by successively removing non-bridges until only bridges remain. (A *non-bridge* means an edge that is not a bridge.)

Now, let us perform this algorithm, but taking care to choose a non-bridge of largest weight (among all non-bridges) at each step. Prove that the result will be a w-minimum spanning tree.

6.2 Solution

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