Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2022: Homework 4
due 2022-04-27 at 11:00 AM
Please solve 3 of the 6 problems!

Darij Grinberg

May 3, 2023

---

1 Exercise 1

1.1 Problem

(a) Let $G = (V, E)$ be a simple graph, and let u and v be two distinct vertices of G that are not adjacent. Let $n = |V|$. Assume that $\deg u + \deg v \geq n$. Let $G' = (V, E \cup \{uv\})$ be the simple graph obtained from G by adding a new edge uv. Assume that G' has a hamc. Prove that G has a hamc.

(b) Does this remain true if we replace "hamc" by "hamp"?

1.2 Solution

...

---

2 Exercise 2

2.1 Problem

Let D be a multidigraph with at least one vertex. Prove the following:

(a) If each vertex v of D satisfies $\deg^+ v > 0$, then D has a cycle.

(b) If each vertex v of D satisfies $\deg^+ v = \deg^- v = 1$, then each vertex of D belongs to exactly one cycle of D. Here, two cycles are considered to be identical if one can be obtained from the other by cyclic rotation.

2.2 Solution

...

---

3 Exercise 3

3.1 Problem

Prove the directed Euler–Hierholzer theorem:

Let D be a weakly connected multidigraph. Then:

(a) The multidigraph D has an Eulerian circuit if and only if each vertex v of D satisfies $\deg^+ v = \deg^- v$.

(b) The multidigraph D has an Eulerian walk if and only if all but two vertices v of D satisfy $\deg^+ v = \deg^- v$, and the remaining two vertices v satisfy $\left|\deg^+ v - \deg^- v\right| \leq 1$.

3.2 Solution

...

---

4 Exercise 4

4.1 Problem

(a) Let $D = (V, A, \psi)$ be a multidigraph, where $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

If M is any matrix, and if i and j are two positive integers, then $M_{i,j}$ shall denote the (i, j)-th entry of M (that is, the entry of M in the i-th row and the j-th column).

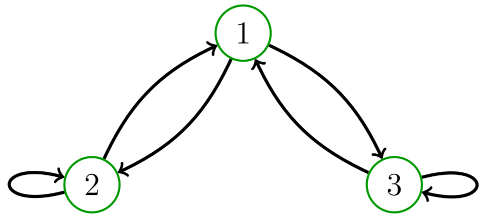
Let C be the $n \times n$-matrix (with real entries) defined by

$$C_{i,j} = (\text{the number of all arcs } a \in A \text{ with source } i \text{ and target } j) \qquad \text{for all } i, j \in V.$$

Let $k \in \mathbb{N}$, and let $i, j \in V$. Prove that $\left(C^k\right)_{i,j}$ equals the number of all walks of D having starting point i, ending point j and length k.

(b) Let E be the following multidigraph:

$E =$

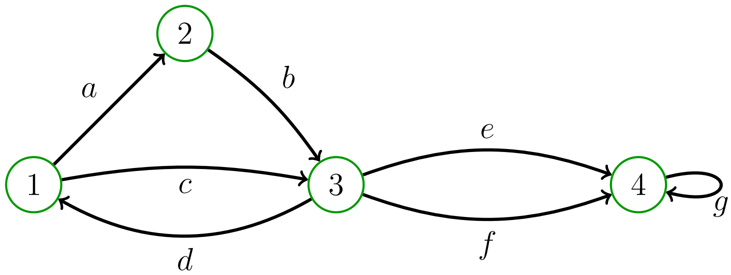


Let $n \in \mathbb{N}$. Compute the number of walks from 1 to 1 having length n.

4.2 Remark

The matrix C in part **(a)** of this problem is known as the *adjacency matrix* of D. For example, if the multidigraph is

$D =$



then its adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The adjacency matrix of a multidigraph D determines D up to the identities of the arcs, and thus is often used as a convenient way to encode a multidigraph.

4.3 Solution

...

---

5 Exercise 5

5.1 Problem

Consider a multidigraph $D = (V, A, \psi)$. Let $\ell(a)$ be a real number for each arc $a \in A$. We view $\ell(a)$ as a measure of "length" of the arc a (essentially modeling how long it takes or how hard it is to get from the source of a to the target of a by walking along this arc). For any walk $\mathbf{w} = (w_0, a_1, w_1, a_2, w_2, \ldots, a_k, w_k)$ of D, we define the *weighted length* of $\mathbf{w}$ to be the sum $\ell(a_1) + \ell(a_2) + \cdots + \ell(a_k)$ of the lengths of all arcs of $\mathbf{w}$. We denote this weighted length by $\ell(\mathbf{w})$.

(Note that if we set $\ell(a) = 1$ for each arc a, then this weighted length is just the usual length of $\mathbf{w}$.)

We are looking for a quick way of finding, for any two vertices u and v of D, the smallest weighted length of a walk from u to v (if such a walk exists), and at least one walk having this smallest weighted length. (This models quite a few real-life optimization problems, if the digraph D and the lengths $\ell(a)$ are chosen appropriately.)

It turns out that the best way to approach this problem is to compute these data for all vertices v simultaneously, while keeping u fixed. Here is one way to organize this:

For any $k \in \mathbb{N}$ any two vertices u and v of D, we define a *k-optimal u-v-walk* to mean a walk from u to v that has length k and has the smallest weighted length among all such walks. If a k-optimal u-v-walk exists, then we denote its length by $d_{\ell,k}(u,v)$. If not, then we define $d_{\ell,k}(u,v)$ to be ∞.

Prove the following:

(a) For any positive integer k and any two vertices u and v, we have

$$d_{\ell,k}(u,v) = \min\{d_{\ell,k-1}(u,w) + d_{\ell,1}(w,v) \mid w \in V\}.$$

(b) For any positive integer k and any two vertices u and v, we have the following: Let $w \in V$ be chosen in such a way that $d_{\ell,k-1}(u,w) + d_{\ell,1}(w,v)$ is minimum. Let $\mathbf{a}$ be a $(k-1)$-optimal u-w-walk. Let $\mathbf{b}$ be a 1-optimal w-v-walk (i.e., a walk consisting of a single arc from w to v, which has to have minimum length among all arcs from w to v). Then, $\mathbf{a} * \mathbf{b}$ is a k-optimal u-v-walk.

Now, assume that the weighted length of each cycle of D is nonnegative.

Prove the following:

(c) If u and v are any two vertices of V, then the smallest weighted length of a walk from u to v equals the smallest weighted length of a **path** from u to v.

(d) Let $n = |V|$. If u and v are any two vertices of V, then the smallest weighted length of a walk from u to v equals $\min\{d_{\ell,k}(u,v) \mid 0 \le k \le n-1\}$.

5.2 Remark

This gives an easy recursive way to find shortest paths between any two vertices of a digraph (and thus also of a graph, because if G is a graph, then the digraph G^{bidir} can be used as a standin for G).

It doesn't take much thought to see that all we need are *signposts*: For any two vertices u and v and any $k \in \mathbb{N}$, we need to know the weighted length and the first arc of a k-optimal u-v-walk. We can imagine this information being written on a signpost at u. Using these signposts, we can find the entire k-optimal u-v-walk by following them successively until we reach v. This is essentially the Bellman–Ford algorithm.

5.3 Solution

...

6 Exercise 6

6.1 Problem

Let D be a simple digraph with n vertices and a arcs. Assume that D has no loops, and that we have $a > n^2/2$. Prove that D has a cycle of length 3.

6.2 Remark

Note that this is both an analogue and a generalization (why?) of Mantel's theorem.

6.3 Solution

...