Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2022: Homework 3
due 2022-04-20 at 11:00 AM
Please solve 3 of the 6 problems!

Darij Grinberg

April 18, 2025

1 Exercise 1

1.1 Problem

Which of the exercises 1, 2, 3, 4, 6 from homework set #1 remain true if "simple graph" is replaced by "multigraph"?

(For each exercise that becomes false, provide a counterexample. For each exercise that remains true, either provide a new solution that works for multigraphs, or argue that the solution we have seen applies verbatim to multigraphs, or derive the multigraph case from the simple graph case.)

1.2 Solution

...

2 Exercise 2

2.1 Problem

Let G be a multigraph with at least one vertex. Let $d > 2$ be an integer. Assume that $\deg v > 2$ for each vertex v of G. Prove that G has a cycle whose length is not divisible by d.

2.2 Solution

...

---

3 Exercise 3

3.1 Problem

Let $G = (V, E, \varphi)$ be a multigraph that has no loops.

If $e \in E$ is an edge that contains a vertex $v \in V$, then we let e/v denote the endpoint of e distinct from v. (If e is a loop, then this is understood to mean v itself.)

For each $v \in V$, we define a rational number q_v by

$$q_v = \sum_{\substack{e \in E; \\ v \in \varphi(e)}} \frac{\deg(e/v)}{\deg v}.$$

(Note that the denominator $\deg v$ on the right hand side is nonzero whenever the sum is nonempty!)

(Thus, q_v is the average degree of the neighbors of v, weighted with the number of edges that join v to the respective neighbors. If v has no neighbors, then $q_v = 0$.)

Prove that

$$\sum_{v \in V} q_v \geq \sum_{v \in V} \deg v.$$

(In other words, in a social network, your average friend has, on average, more friends than you do!)

3.2 Hint

Any two positive reals x and y satisfy $\frac{x}{y} + \frac{y}{x} \geq 2$. Why, and how does this help?

3.3 Solution

...

---

4 Exercise 4

4.1 Problem

Let n and k be two integers such that $n > k > 0$. Define the simple graph $Q_{n,k}$ as follows: Its vertices are the bitstrings $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$; two such bitstrings are adjacent if and only if they differ in exactly k bits[1]. (Thus, $Q_{n,1}$ is the n-hypercube graph Q_n.)

(a) Does $Q_{n,k}$ have a hamc[2] when k is even?

(b) Does $Q_{n,k}$ have a hamc when k is odd?

4.2 Remark

One way to approach part **(b)** is by identifying the set $\{0,1\}$ with the field $\mathbb{F}_2$ with two elements. The bitstrings $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ thus become the size-n row vectors in the $\mathbb{F}_2$-vector space $\mathbb{F}_2^n$. Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors of $\mathbb{F}_2^n$ (so that e_i has a 1 in its i-th position and zeroes everywhere else). Then, two vectors are adjacent in the n-hypercube graph Q_n (resp. in the graph $Q_{n,k}$) if and only if their difference is one of the standard basis vectors (resp., a sum of k distinct standard basis vectors). Try to use this to find a graph isomorphism from Q_n to a subgraph of $Q_{n,k}$.

4.3 Solution

...

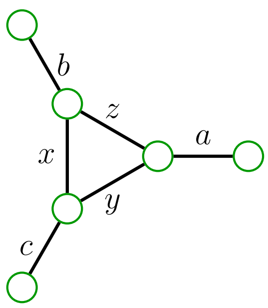
---

5 Exercise 5

5.1 Problem

Let $G = (V, E, \varphi)$ be a connected multigraph with $2m$ edges, where $m \in \mathbb{N}$. A set $\{e, f\}$ of two distinct edges will be called a *friendly couple* if e and f have at least one endpoint in common. Prove that the edge set of G can be decomposed into m disjoint[3] friendly couples (i.e., there exist m disjoint friendly couples $\{e_1, f_1\}, \{e_2, f_2\}, \ldots, \{e_m, f_m\}$ such that $E = \{e_1, f_1, e_2, f_2, \ldots, e_m, f_m\}$).

[**Example:** Here is a graph with an even number of edges:



[1] In other words: Two vertices $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are adjacent if and only if the number of $i \in \{1, 2, \ldots, n\}$ satisfying $a_i \neq b_i$ equals k.

[2] Recall that "hamc" is short for "Hamiltonian cycle".

[3] "Disjoint" means "disjoint as sets" – i.e., having no edges in common.

One possible decomposition into disjoint friendly pairs is $\{a, y\}, \{b, z\}, \{c, x\}$.]

5.2 Hint

Induct on $|E|$. Pick a vertex v of degree > 1 and consider the components of $G \setminus v$.

5.3 Solution

...

---

6 Exercise 6

6.1 Problem

Let $n \geq 0$. Let $d_1, d_2, \ldots, d_n$ be n nonnegative integers such that $d_1 + d_2 + \cdots + d_n$ is even.

(a) Prove that there exists a multigraph G with vertex set $\{1, 2, \ldots, n\}$ such that all $i \in \{1, 2, \ldots, n\}$ satisfy $\deg i = d_i$.

(b) A multigraph is said to be *loopless* if it has no loops. Prove that there exists a loopless multigraph G with vertex set $\{1, 2, \ldots, n\}$ such that all $i \in \{1, 2, \ldots, n\}$ satisfy $\deg i = d_i$ if and only if each $i \in \{1, 2, \ldots, n\}$ satisfies the inequality

$$\sum_{\substack{j \in \{1,2,\ldots,n\}; \\ j \neq i}} d_j \geq d_i. \qquad (1)$$

6.2 Remark

The inequality (1) is the "n-gon inequality": It is equivalent to the existence of a (possibly degenerate) n-gon with sidelengths $d_1, d_2, \ldots, d_n$.

6.3 Solution

...
