Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2022: Homework 1
due 2022-04-06 at 11:00 AM
Please solve 4 of the 8 problems!

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1 Exercise 1

1.1 Problem

Let $G = (V, E)$ be a simple graph. Set $n = |V|$. Prove that we can find some edges $e_1, e_2, \ldots, e_k$ of G and some triangles $t_1, t_2, \ldots, t_\ell$ of G such that $k + \ell \leq n^2/4$ and such that each edge $e \in E \setminus \{e_1, e_2, \ldots, e_k\}$ is a subset of (at least) one of the triangles $t_1, t_2, \ldots, t_\ell$.

1.2 Remark

This is a generalization of Mantel's theorem (because if G has no triangles, then ℓ must be 0, and thus $e_1, e_2, \ldots, e_k$ must be all edges of G, so that we conclude that $|E| = k \leq k + \ell \leq n^2/4$).

1.3 Solution

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2 Exercise 2

2.1 Problem

Let G be a simple graph with n vertices and k edges, where $n > 0$. Prove that G has at least $\frac{k}{3n}\left(4k - n^2\right)$ triangles.

2.2 Hint

First argue that for any edge vw of G, the total number of triangles that contain v and w is at least $\deg v + \deg w - n$. Then, use the inequality $n\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \geq \left(a_1 + a_2 + \cdots + a_n\right)^2$, which holds for any n real numbers $a_1, a_2, \ldots, a_n$. (This is a particular case of the Cauchy–Schwarz inequality or the Chebyshev inequality or the Jensen inequality – pick your favorite!)

2.3 Remark

This, too, is a generalization of Mantel's theorem: If $k > n^2/4$, then $\frac{k}{3n}\left(4k - n^2\right) > 0$, so the exercise entails that G has at least one triangle.

2.4 Solution

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3 Exercise 3

3.1 Problem

Let n be a positive integer. Let S be a simple graph with $2n$ vertices. Prove that S has two distinct vertices that have an even number of common neighbors.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $n \geq 2$ be an integer. Let G be a connected simple graph with n vertices.

(a) Describe G if the degrees of the vertices of G are $1, 1, 2, 2, \ldots, 2$ (exactly two 1's and $n - 2$ many 2's).

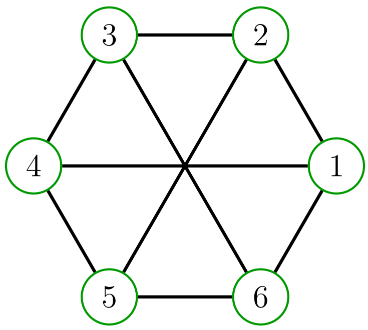
(b) Describe G if the degrees of the vertices of G are $1, 1, \ldots, 1, n - 1$.

(c) Describe G if the degrees of the vertices of G are $2, 2, \ldots, 2$.

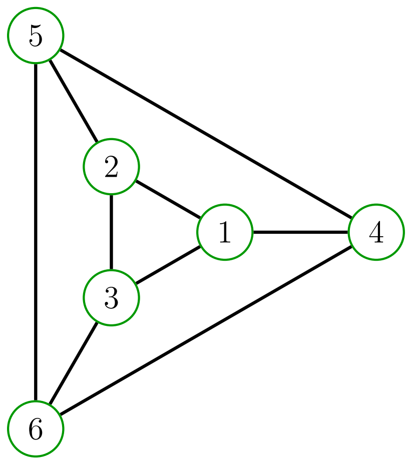
Here, to "describe" G means to explicitly determine (with proof) a graph that is isomorphic to G.

4.2 Remark

The situations in this exercise are, in a sense, exceptional. Typically, the degrees of the vertices of a connected graph do not uniquely determine the graph up to isomorphism. For example, the two connected graphs



and



are not isomorphic[1], but have the same degrees (namely, each vertex of either graph has degree 3).

4.3 Solution

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5 Exercise 5

5.1 Problem

A simple graph $G = (V, E)$ is said to be *optibip* (short for "optimal bipartite") if the set V can be partitioned into two subsets V_1 and V_2 such that $|V_1| - |V_2| \in \{0, 1\}$ and such that $E = \{v_1 v_2 \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}$. (The word "partitioned" means that V_1 and V_2 are two disjoint sets whose union is V.)

Let $G = (V, E)$ be a simple graph with n vertices and k edges. Prove that the following two statements are equivalent:

1. The graph G has no triangles and satisfies $k = \lfloor n^2/4 \rfloor$.

2. The graph G is optibip.

[1] The easiest way to see this is to observe that the second graph has a triangle (i.e., three distinct vertices that are mutually adjacent), while the first graph does not.

5.2 Remark

This characterizes when equality holds in (the contrapositive of) Mantel's theorem.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let G be a simple graph with n vertices. Assume that each vertex of G has at least one neighbor.

A *matching* of G shall mean a set F of edges of G such that no two edges in F have a vertex in common. Let m be the largest size of a matching of G.

An *edge cover* of G shall mean a set F of edges of G such that each vertex of G is contained in at least one edge $e \in F$. Let c be the smallest size of an edge cover of G.

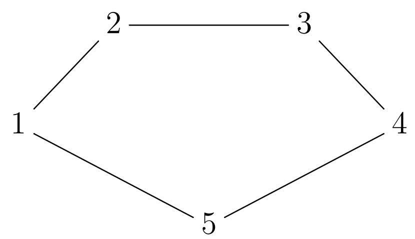
Prove that $c + m = n$.

6.2 Hint

Prove that $c \leq n - m$ and that $m \geq n - c$.

6.3 Remark

Let G be the graph



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Then, $\{12, 34\}$ is a matching of G of largest possible size (why?), whereas $\{12, 34, 25\}$ is an edge cover of G of smallest possible size (why?). So the exercise says that $2 + 3 = 5$ here, which is indeed true.

6.4 Solution

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7 Exercise 7

7.1 Problem

Let $G = (V, E)$ be a simple graph.

An edge $e = \{u, v\}$ of G will be called *odd* if the number $\deg u + \deg v$ is odd.

Prove that the number of odd edges of G is even.

7.2 Hint

One solution uses modular arithmetic. Note in particular that $m^2 \equiv m \mod 2$ for every integer m. (There is also a solution without all of this.)

7.3 Solution

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8 Exercise 8

8.1 Problem

Let $n \geq 2$ be an integer. Let G be a simple graph with n vertices.

(a) Describe G if the degrees of the vertices of G are $1, 1, \ldots, 1, n-1$.

(b) Let a and b be two positive integers such that $a + b = n$. Describe G if the degrees of the vertices of G are $1, 1, \ldots, 1, a, b$.

Here, to "describe" G means to explicitly determine (with proof) a graph that is isomorphic to G.

8.2 Remark

This is a variation on Exercise 4 above. Note that G is not required to be connected here.

References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`