Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2022: Midterm 3

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due date: **Monday, 2022-12-12** at 11:59 PM on gradescope.

Please solve **3 of the 7 exercises**!

Collaboration is **not allowed** on this midterm!

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1 Exercise 1

1.1 Problem

Let n be a positive integer.

(a) Compute the Stirling number of the 1st kind $\begin{bmatrix} n \\ n-1 \end{bmatrix}$. (Express the result without summation signs.)

(b) Let H_{n-1} denote the $(n-1)$-st *harmonic number*, defined as the sum $\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n-1}$. Prove that

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1}.$$

1.2 Solution

[...]

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Prove that

$$\sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix} \binom{p}{k} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}.$$

2.2 Hint

Substitute $X + 1$ for X in the polynomial identity

$$X (X+1)(X+2) \cdots (X+n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} X^k.$$

Then expand the right hand side and multiply by X.

2.3 Solution

[...]

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. How many (ordered) pairs (A, B) of two nonempty subsets of $[n]$ have the property that $\min A > |B|$ and $\min B > |A|$?

[**Example:** If $n = 7$, then the pair $(\{3,5,7\}, \{4,5\})$ qualifies, since $\min\{3,5,7\} = 3 > 2 = |\{4,5\}|$ and $\min\{4,5\} = 4 > 3 = |\{3,5,7\}|$. However, the pair $(\{2,5,7\}, \{4,5\})$ does not qualify, since $\min\{2,5,7\} \leq |\{4,5\}|$.]

3.2 Solution

[...]

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4 Exercise 4

4.1 Problem

Let n and k be two positive integers such that $k > 1$.

Let u be the # of compositions α of n such that each entry of α is congruent to 1 modulo k.

Let v be the # of compositions β of $n + k - 1$ such that each entry of β is $\geq k$.

Let w be the # of compositions γ of $n-1$ such that each entry of γ is either 1 or k.
Prove that $u = v = w$.

4.2 Remark

The equality $u = w$ was Exercise 2 on homework set #5, and can be used without proof now.

4.3 Solution

[...]

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5 Exercise 5

5.1 Problem

Let n be an integer such that $n > 1$.

(a) Find the # of all permutations $\sigma \in S_n$ such that $\sigma(1) = \sigma^{-1}(1)$.

(b) Find the # of all permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma^{-1}(1)$.

5.2 Hint

For part **(b)**, relate the sizes of the three sets

$$A_< := \left\{\sigma \in S_n \mid \sigma(1) < \sigma^{-1}(1)\right\},$$
$$A_= := \left\{\sigma \in S_n \mid \sigma(1) = \sigma^{-1}(1)\right\},$$
$$A_> := \left\{\sigma \in S_n \mid \sigma(1) > \sigma^{-1}(1)\right\}$$

to one another.

5.3 Solution

[...]

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$, and let $k \in [n]$. For any permutation $\sigma \in S_n$, we let $m_k(\sigma)$ be the # of orbits of σ that have size k (that is, the # of k-cycles in the disjoint cycle decomposition of σ).

Prove that

$$\sum_{\sigma \in S_n} m_k(\sigma) = \frac{n!}{k}.$$

6.2 Remark (and perhaps a hint)

In other words, this claims that the expected # of orbits of size k in a permutation $\sigma \in S_n$ is $\frac{1}{k}$. In particular, for $k = 1$, this shows that the expected # of fixed points of a permutation $\sigma \in S_n$ is 1 (since the fixed points of σ correspond to the orbits of size 1). See, e.g., [17f-hw7s, solution to Exercise 2] or [20f, §7.4.3] for a proof of this particular case.

6.3 Solution

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7 Exercise 7

7.1 Problem

Let $a, b, c \in \mathbb{N}$. Show that

$$\sum_{k \in \mathbb{Z}} (-1)^k \binom{2a}{a+k} \binom{2b}{b+k} \binom{2c}{c+k} = \frac{(a+b+c)! \cdot (2a)! \cdot (2b)! \cdot (2c)!}{a!b!c! \cdot (b+c)! \cdot (c+a)! \cdot (a+b)!}.$$

7.2 Hint

The proof is quite short using what was done in class.

7.3 Solution

[...]

References

[17f-hw7s] Darij Grinberg, *UMN Fall 2017 Math 4990 homework set #7 with solutions*, http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw7os.pdf

[20f] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 22 March 2021. http://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 13 September 2022. http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf Also available on the mirror server http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf