Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2022: Midterm 2

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due date: **Monday, 2022-11-28** at 11:59 PM on gradescope.

Please solve **3 of the 6 exercises**!

Collaboration is **not allowed** on this midterm!

1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Prove that

$$\left(\# \text{ of all } (m_1, m_2, \ldots, m_k) \in \{0,1,2,3\}^k \text{ such that } m_1 + m_2 + \cdots + m_k = n\right) = \sum_{j=0}^{k} \binom{k}{j} \binom{k}{n-2j}.$$

1.2 Hint

How can an element of $\{0,1,2,3\}$ be made out of two elements of $\{0,1\}$?

1.3 Solution

[...]

2 Exercise 2

2.1 Problem

Let n be a positive integer. Compute the Stirling numbers of the 2nd kind $\left\{ \begin{matrix} n \\ n \end{matrix} \right\}$, $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\}$ and $\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\}$.

(In each case, express the result without summation signs.)

2.2 Solution

[...]

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3 Exercise 3

3.1 Problem

Let n and m be two positive integers. An (n, m)-*stairway* will mean a map $f : \mathbb{Z} \to \mathbb{Z}$ that is weakly increasing (i.e., satisfies $f(x) \leq f(x+1)$ for each integer x) and satisfies $f(0) = 0$ and

$$f(x+n) = f(x) + m \qquad \text{for each integer } x.$$

Find the # of all (n, m)-stairways.

3.2 Remark

An example of an (n, m)-stairway is the map

$$\mathbb{Z} \to \mathbb{Z}, \qquad x \mapsto \lfloor mx/n \rfloor .$$

3.3 Solution

[...]

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4 Exercise 4

4.1 Problem

Recall Exercise 1 in §2.11 of Lecture 23 (the exercise about n people and k tasks). Now, modify this exercise by allowing tasks to remain unclaimed (i.e., there might be 0 people working on some task).

Solve all three parts **(a)**, **(b)** and **(c)** of the exercise in this modified form. (In part **(b)**, an unclaimed task is allowed to remain leaderless.)[1]

[1] The answer to part **(b)** will involve a summation sign; I don't think it can be avoided.

4.2 Solution

[...]

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5 Exercise 5

5.1 Problem

Let n and k be two positive integers.

A set partition $\{S_1, S_2, \ldots, S_k\}$ of $[n]$ will be called *lacunar* if all of its parts $S_1, S_2, \ldots, S_k$ are lacunar (i.e., if no two adjacent integers belong to the same part of it).

Prove that the # of lacunar set partitions of $[n]$ into k parts is $\left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$.

5.2 Hint

This suggests that there should be a bijection between lacunar set partitions of $[n]$ into k parts and (arbitrary) set partitions of $[n-1]$ into $k-1$ parts. Such bijections indeed exist, but are not easy to find.

It is easier to reduce the problem to counting certain surjections. Namely, let $\operatorname{lur}(n,k)$ be the # of surjections $f : [n] \to [k]$ that never take the same value twice in succession (i.e., that satisfy $f(i) \neq f(i+1)$ for all $i \in [n-1]$). Can you find a recursion for $\operatorname{lur}(n,k)$? Can you use this to show that $\operatorname{lur}(n,k) = k \operatorname{sur}(n-1, k-1)$? And how do these surjections relate to lacunar set partitions?

5.3 Solution

[...]

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Prove that

$$\operatorname{sur}(n,k) = \sum_{\substack{(a_1, a_2, \ldots, a_k) \text{ is a composition} \\ \text{of } n \text{ into } k \text{ parts}}} 1^{a_1} 2^{a_2} \cdots k^{a_k}.$$

6.2 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 13 September 2022. http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf Also available on the mirror server http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf