Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2022: Midterm 1

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due date: **Friday, 2022-11-04** at 11:59 PM on gradescope.

Please solve **4 of the 8 exercises**!

Collaboration is **not allowed** on this midterm!

1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\prod_{i=0}^{n} \binom{2i}{i}^2 = 2^n \prod_{i=0}^{n} \left(\binom{n+i}{i} \binom{n}{i} \right).$$

1.2 Solution

[...]

2 Exercise 2

2.1 Problem

Let n be a positive integer, and let $m \in \mathbb{N}$. Let $p_1, p_2, \ldots, p_n \in \mathbb{N}$ be fixed. Prove that

$$\sum_{\substack{(a_1,a_2,\ldots,a_n)\in\mathbb{N}^n;\\ a_1+a_2+\cdots+a_n=m}} \binom{a_1}{p_1}\binom{a_2}{p_2}\cdots\binom{a_n}{p_n} = \binom{m+n-1}{p_1+p_2+\cdots+p_n+n-1}.$$

2.2 Hint

You are free to use results stated in the notes, such as the Chu–Vandermonde identity.

2.3 Solution

[...]

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. Give a closed-form expression (no summation signs) for $\sum\limits_{k=0}^{n} \left\lfloor \dfrac{k}{2} \right\rfloor \dbinom{n}{k}$.

3.2 Solution

[...]

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4 Exercise 4

4.1 Problem

A set S of integers will be called *almost-lacunar* if there exists exactly one $i \in S$ satisfying $i+1 \in S$.

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Find the # of k-element almost-lacunar subsets of $[n]$.

4.2 Solution

[...]

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5 Exercise 5

5.1 Problem

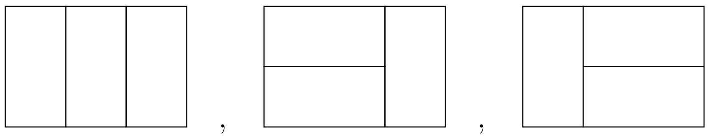
For each $n \in \mathbb{N}$, define the polynomial

$$T_n(x) := \sum_{T \text{ is a domino tiling of } R_{n,2}} (-1)^{h(T)} (2x)^{v_+(T)} x^{v_1(T)}$$

in a single indeterminate x with integer coefficients, where

- $h(T) \in \mathbb{N}$ denotes the # of horizontal dominos in the bottom row of T;
- $v_+(T) \in \mathbb{N}$ denotes the # of vertical dominos in columns $2, 3, \ldots, n$ of T;
- $v_1(T) \in \{0, 1\}$ denotes the # of vertical dominos in column 1 of T.

For instance, because of the three domino tilings



of $R_{3,2}$, we have

$$\begin{aligned} T_3(x) &= (-1)^0 (2x)^2 x^1 + (-1)^1 (2x)^1 x^0 + (-1)^1 (2x)^0 x^1 \\ &= 4x^3 - 2x - x = 4x^3 - 3x. \end{aligned}$$

(a) Prove that $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for any $n \geq 2$.

(b) Prove that $\cos(n\alpha) = T_n(\cos\alpha)$ for any $n \in \mathbb{N}$ and any angle α.

(c) Prove that $\dfrac{z^n + z^{-n}}{2} = T_n\left(\dfrac{z + z^{-1}}{2}\right)$ for any $n \in \mathbb{N}$ and any nonzero $z \in \mathbb{C}$.

(d) Prove that $T_m(T_n(w)) = T_{mn}(w)$ for any $m, n \in \mathbb{N}$ and $w \in \mathbb{C}$.

5.2 Remark

The polynomials $T_0, T_1, T_2, \ldots$ are known as the *Chebyshev polynomials of the first kind.* (However, they aren't usually defined in such an explicit way.)

5.3 Hint

You can use trigonometric identities (but not ones that involve the Chebyshev polynomials). Don't forget that the empty rectangle $R_{0,2}$ has a single domino tiling (which is itself an empty set, i.e., contains no dominos).

5.4 Solution

[...]

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} (-2)^k \binom{n}{k} \binom{2n-k}{n-k} = \begin{cases} (-1)^{n/2} \dbinom{n}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

6.2 Solution

[...]

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7 Exercise 7

7.1 Problem

For nonnegative integers x, n, u, and v, set

$$F_{x,n}(u,v) := \sum_{i=0}^{v} \binom{x+u}{n-i} \binom{u+i}{i} \binom{v}{i}.$$

Prove that

$$F_{x,n}(u,v) = F_{x,n}(v,u)$$

for any $x, n, u, v \in \mathbb{N}$.

7.2 Solution

[...]

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8 Exercise 8

8.1 Problem

Let k be a positive integer. A set S of integers will be called "la$k$unar" if there exists no $i \in S$ such that $i + k \in S$. (Thus, the lacunar sets are precisely the la1unar sets.)

For any $n \in \mathbb{N}$, let $a_{n,k}$ denote the maximum size of a la$k$unar subset of $[n]$.

(a) Make a table of $a_{n,k}$ for all $1 \leq n \leq 8$ and $1 \leq k \leq 8$.

(b) Prove that $a_{n,k} \leq \dfrac{n+k}{2}$ for all $n \in \mathbb{N}$.

(c) Is this always an equality?

(d) Find the sequence of the $a_{n,2}$ in the OEIS.

(e)* Prove a "rule" (not necessarily stated as a formula) that makes the computation of $a_{n,k}$ easy and quick.

8.2 Hint

If you are using SageMath, note that `Subsets(n, k)` yields the set of all k-element subsets of $[n]$.

8.3 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 13 September 2022. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`