Math 222 Fall 2022, Lecture 30: Generating functions

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

6. A brief introduction to generating functions

We have previously seen how the concept of polynomials can be used to prove combinatorial identities. Namely, we have used the polynomial identity trick (Lecture 14, Corollary 2.2.5) to generalize identities from $x \in \mathbb{N}$ to $x \in \mathbb{R}$ (under certain conditions). The underlying idea is that if two polynomials P and Q (in a single indeterminate, with real coefficients) satisfy $P(x) = Q(x)$ for all $x \in \mathbb{N}$, then they are identical as polynomials (and thus $P(x) = Q(x)$ also holds for all $x \in \mathbb{R}$).

Today I will show another way to use polynomials – this time, without computing any of their values. That is, we will just regard polynomials as "formal expressions" of the form $a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$, which are added and subtracted coefficientwise and multiplied by the standard rule (distributivity and $(aX^n) \cdot (bX^m) = abX^{n+m}$). We will not substitute any actual numbers for the indeterminate X, but merely manipulate polynomials algebraically and then extract certain coefficients. This doesn't sound like a particularly deep or useful activity, but it turns out to give short proofs to various binomial identities that are otherwise far from trivial.

This activity is actually a simple instance of the technique of "**generating functions**". In general, a generating function doesn't have to be a polynomial, but can be any **formal power series** (i.e., roughly speaking, an "infinite polynomial", or "polynomial with infinitely many nonzero coefficients", such as $1X + 2X^2 + 3X^3 + \cdots = \sum_{k \in \mathbb{N}} kX^k$). A proper introduction into this technique would work at this general level (such introductions can be found, e.g., in [21s, Chapter 3] or [Loehr11, Chapter 7 (in the 1st edition)][1] or [Sambal22]; see also [Niven69] for a short overview). Here, however, we will content ourselves with polynomials.

6.1. Notations and the binomial formula revisited

Convention 6.1.1. In this whole chapter, X shall denote an indeterminate, and we will be working mostly with polynomials in a single indeterminate X with rational coefficients.

[1] I am deliberately citing the 1st edition of this text, since the 2nd edition does a worse job properly defining formal power series.

Definition 6.1.2. Let P be a polynomial in the indeterminate X. Let $n \in \mathbb{N}$. Then, the X^n-coefficient of P (that is, the coefficient before X^n in P) will be denoted by $[X^n]\, P$.

(Yes, this is one more meaning for square brackets we are introducing. Fortunately, it does not clash with the notation $[n]$ for the set $\{1, 2, \ldots, n\}$, nor with the notation $[\mathcal{A}]$ for the truth value of $\mathcal{A}$.)

For example,

$$\left[X^3\right]\left((1+X)^5\right) = \left[X^3\right]\left(1+5X+10X^2+10X^3+5X^4+1\right) = 10 \qquad \text{and}$$
$$\left[X^8\right]\left((1+X)^5\right) = 0.$$

Let us recall how two polynomials are multiplied:

Proposition 6.1.3. Let

$$P = a_0X^0 + a_1X^1 + a_2X^2 + \cdots = \sum_{m\in\mathbb{N}} a_mX^m \qquad \text{and}$$
$$Q = b_0X^0 + b_1X^1 + b_2X^2 + \cdots = \sum_{m\in\mathbb{N}} b_mX^m$$

be two polynomials.[2] Then,

$$PQ = \sum_{m\in\mathbb{N}} \left(\sum_{i=0}^{m} a_ib_{m-i}\right) X^m. \tag{1}$$

In other words, for each $m \in \mathbb{N}$, the coefficient of X^m in the polynomial PQ is given by

$$[X^m]\,(PQ) = \sum_{i=0}^{m} a_ib_{m-i} = \sum_{i=0}^{m} \left(\left[X^i\right] P\right) \cdot \left(\left[X^{m-i}\right] Q\right). \tag{2}$$

Depending on how you define polynomials, Proposition 6.1.3 is either the definition of the product PQ or an easy consequence of that definition. It is precisely what is obtained when you multiply the equalities $P = \sum_{m\in\mathbb{N}} a_mX^m =$

[2] The sums are infinite, but are supposed to contain only finitely many nonzero addends. For example, when we write $P = \sum_{m\in\mathbb{N}} a_mX^m$, we assume that all sufficiently large $m \in \mathbb{N}$ satisfy $a_m = 0$. (If we drop this requirement, then we end up with **formal power series** rather than polynomials; for now let's not go there.)

$\sum\limits_{i\in\mathbb{N}} a_i X^i$ and $Q = \sum\limits_{m\in\mathbb{N}} b_m X^m = \sum\limits_{j\in\mathbb{N}} b_j X^j$ and expand (using distributivity):

$$PQ = \left(\sum_{i\in\mathbb{N}} a_i X^i\right)\left(\sum_{j\in\mathbb{N}} b_j X^j\right) = \underbrace{\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}}_{=\sum\limits_{(i,j)\in\mathbb{N}\times\mathbb{N}}} \underbrace{a_i X^i b_j X^j}_{=a_i b_j X^{i+j}}$$

$$= \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_i b_j X^{i+j} = \sum_{m\in\mathbb{N}} \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=m}} a_i b_j \underbrace{X^{i+j}}_{\substack{=X^m\\ \text{(since } i+j=m\text{)}}}$$

$$\left(\begin{array}{c}\text{here, we have split up the sum}\\ \text{according to the value of } i+j\end{array}\right)$$

$$= \sum_{m\in\mathbb{N}} \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=m}} a_i b_j X^m = \sum_{m\in\mathbb{N}} \underbrace{\left(\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=m}} a_i b_j\right)}_{\substack{=\sum\limits_{i=0}^{m} a_i b_{m-i}\\ \text{(since the pairs } (i,j)\in\mathbb{N}\times\mathbb{N} \text{ satisfying } i+j=m\\ \text{are precisely the pairs } (0,m),\ (1,m-1),\ \ldots,\ (m,0),\\ \text{that is, the pairs } (i,m-i) \text{ for } i\in\{0,1,\ldots,m\}\text{)}}} X^m$$

$$= \sum_{m\in\mathbb{N}} \left(\sum_{i=0}^{m} a_i b_{m-i}\right) X^m.$$

This proves (1). The equality (2) easily follows from this.

Addition of polynomials is simpler: It is coefficientwise. That is, for any two polynomials P and Q and any $n \in \mathbb{N}$, we have

$$[X^n](P+Q) = ([X^n]P) + ([X^n]Q). \tag{3}$$

Recall the binomial formula (Lecture 7, Theorem 1.3.19), which states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{4}$$

for any $n \in \mathbb{N}$ and any real numbers x and y. Since we proved this formula by formal algebraic manipulations, we see instantaneously that it holds not only when x and y are real numbers, but also more generally when x and y are two polynomials (and the same proof that we gave applies verbatim in this case). This has a few useful consequences:

Lemma 6.1.4. Let $n \in \mathbb{N}$. Then:

(a) We have

$$(1+X)^n = \sum_{m \in \mathbb{N}} \binom{n}{m} X^m.$$

(b) We have

$$(1-X)^n = \sum_{m \in \mathbb{N}} (-1)^m \binom{n}{m} X^m.$$

(c) We have

$$\left(1-X^2\right)^n = \sum_{m \in \mathbb{N}} (-1)^m \binom{n}{m} X^{2m}.$$

Proof. **(a)** We have seen that the binomial formula (4) holds for any polynomials x and y. Hence, we can apply this formula to $x = X$ and $y = 1$. We thus obtain[3]

$$\begin{aligned}(X+1)^n &= \sum_{k=0}^{n} \binom{n}{k} X^k \underbrace{1^{n-k}}_{=1} = \sum_{k=0}^{n} \binom{n}{k} X^k \\ &= \sum_{m=0}^{n} \binom{n}{m} X^m \overset{0}{=} \sum_{m \in \mathbb{N}} \binom{n}{m} X^m\end{aligned}$$

(since every integer $m > n$ satisfies $\binom{n}{m} = 0$ and therefore $\binom{n}{m} X^m = 0X^m = 0$). Since $X + 1 = 1 + X$, this proves Lemma 6.1.4 **(a)**.

(b) This is similar to part **(a)**. Namely, apply the binomial formula (4) to $x = -X$ and $y = 1$. We thus obtain

$$\begin{aligned}(-X+1)^n &= \sum_{k=0}^{n} \binom{n}{k} \underbrace{(-X)^k}_{=(-1)^k X^k} \underbrace{1^{n-k}}_{=1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k X^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} X^k \\ &= \sum_{m=0}^{n} (-1)^m \binom{n}{m} X^m \overset{0}{=} \sum_{m \in \mathbb{N}} (-1)^m \binom{n}{m} X^m.\end{aligned}$$

Since $-X + 1 = 1 - X$, this proves Lemma 6.1.4 **(b)**.

(c) This is similar to part **(b)**. Namely, apply the binomial formula (4) to

[3] Recall that the symbol "$\overset{0}{=}$" means "equal because the two sums differ only in addends which are zero". In other words, we write "$a \overset{0}{=} b$" if a and b are two sums that differ only in addends that are zero. As we know, any two such sums a and b are necessarily equal.

$x = -X^2$ and $y = 1$. We thus obtain

$$\left(-X^2+1\right)^n = \sum_{k=0}^{n} \binom{n}{k} \underbrace{\left(-X^2\right)^k}_{=(-1)^k X^{2k}} \underbrace{1^{n-k}}_{=1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k X^{2k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} X^{2k}$$
$$= \sum_{m=0}^{n} (-1)^m \binom{n}{m} X^{2m} \overset{0}{=} \sum_{m \in \mathbb{N}} (-1)^m \binom{n}{m} X^{2m}.$$

Since $-X^2+1 = 1-X^2$, this proves Lemma 6.1.4 **(c)**. □

6.2. A new proof of the Chu–Vandermonde identity

Now let us see some applications.

Recall the Chu–Vandermonde identity (Lecture 15, Theorem 2.3.1), which states that

$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} \tag{5}$$

for all $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. (We have renamed x and y as a and b here.)

Here is a new proof of this identity:

New proof of the Chu–Vandermonde identity. As in Lecture 15, we WLOG assume that $a, b \in \mathbb{N}$ (since we can derive the general case from this case using the "polynomial identity trick").

Now, Lemma 6.1.4 **(a)** yields

$$(1+X)^a = \sum_{m \in \mathbb{N}} \binom{a}{m} X^m \qquad \text{and} \qquad (1+X)^b = \sum_{m \in \mathbb{N}} \binom{b}{m} X^m.$$

Hence, the formula (1) for the product of two polynomials (applied to $P = (1+X)^a$, $a_m = \binom{a}{m}$, $Q = (1+X)^b$ and $b_m = \binom{b}{m}$) yields

$$(1+X)^a \cdot (1+X)^b = \sum_{m \in \mathbb{N}} \left(\sum_{i=0}^{m} \binom{a}{i} \binom{b}{m-i} \right) X^m.$$

Hence, the coefficient of X^n in the polynomial $(1+X)^a \cdot (1+X)^b$ is

$$\sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k}.$$

On the other hand, the polynomial $(1+X)^a \cdot (1+X)^b$ can be rewritten as follows:

$$(1+X)^a \cdot (1+X)^b = (1+X)^{a+b} = \sum_{m \in \mathbb{N}} \binom{a+b}{m} X^m$$

(by Lemma 6.1.4 **(a)**). Hence, the coefficient of X^n in this polynomial is $\binom{a+b}{n}$.

Now, we have computed this coefficient in two different ways, obtaining $\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}$ from our first way and $\binom{a+b}{n}$ from our second. But of course, the answers have to be equal. Thus, we obtain

$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}.$$

This proves the Chu–Vandermonde identity. □

Note the strategy that we followed in our above argument: We have proved an equality between two numbers (in our case, (5)) by

- identifying these two numbers as corresponding coefficients of two polynomials (in our case, $(1+X)^a \cdot (1+X)^b$ and $(1+X)^{a+b}$)
- and then showing that the two polynomials are equal.

The argument looks almost magical, seeing how it transformed the trivial-looking equality $(1+X)^a \cdot (1+X)^b = (1+X)^{a+b}$ into the nontrivial Chu–Vandermonde identity (5). This illustrates the power of polynomials (and, more generally, generating functions). A polynomial is not just a convenient "package" for its coefficients, but also provides ways to manipulate them "as a collective" that are not easily available when one works with single coefficients by themselves. A simple manipulation of polynomials can thus "encode" rather complicated arguments on the level of single coefficients.

To remain fully honest (and dispel the magic somewhat), I should mention that the trivial-looking equality $(1+X)^a \cdot (1+X)^b = (1+X)^{a+b}$ is not **entirely** trivial: It is a consequence of the "law of exponents" $P^a P^b = P^{a+b}$, which in turn is proved (by induction on a) using associativity of multiplication. Thus, our proof tacitly relied on the fact that the multiplication of polynomials is associative. This fact is easy to prove (see [21s, proof of Theorem 3.2.6] for a proof), but not completely obvious (its proof requires a bit of sum manipulation). So the above proof is not entirely effortless, at least if you count in the effort needed to prove this fact.

The magic of the above proof is thus not one of immaculate conception. Rather, the concept of a polynomial has allowed a simple result (such as associativity of multiplication for polynomials) to cast a long shadow. This kind of "amplification" of results is a hallmark of abstract algebra.

6.3. The alternating Chu–Vandermonde identity

So we have learnt that simple (even obvious) identities between polynomials can turn into nontrivial identities between their coefficients if you focus on a

single coefficient. This can often be reverse-engineered: When you want to prove a nontrivial identity between two numbers, you might wonder whether these two numbers could be the X^n-coefficients of two equal polynomials whose equality is much more obvious.

This is the technique of **generating functions**. In its general form, instead of using polynomials, it uses formal power series (essentially "polynomials" that can have infinitely many nonzero coefficients). For the examples we shall explore in today's lecture, polynomials will suffice.

Let us try our hands at another application of the same technique. This time, we shall prove a binomial identity that we haven't already proved in a different way:

Theorem 6.3.1 ("alternating Chu–Vandermonde identity")**.** Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then,

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{x}{n-k} = \begin{cases} (-1)^{n/2} \binom{x}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since both sides are polynomial functions in x, it suffices to prove this equality for $x \in \mathbb{N}$ (by the "polynomial identity trick" from Lecture 14). So let us WLOG assume that $x \in \mathbb{N}$, and let us rename x as a. Thus, we must prove that

$$\sum_{k=0}^{n} (-1)^k \binom{a}{k} \binom{a}{n-k} \stackrel{?}{=} \begin{cases} (-1)^{n/2} \binom{a}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

We try to prove this in the same way as we just proved the Chu–Vandermonde identity: We interpret both sides as the X^n-coefficients of polynomials, and we show that the polynomials are equal.

What is the polynomial whose X^n-coefficient is

$$\begin{cases} (-1)^{n/2} \binom{a}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

for each $n \in \mathbb{N}$? I claim that this polynomial is $\left(1 - X^2\right)^a$. Indeed, Lemma

6.1.4 **(c)** yields

$$\begin{aligned}\left(1-X^2\right)^a &= \sum_{m\in\mathbb{N}} (-1)^m \binom{a}{m} X^{2m} \\ &= \binom{a}{0}X^0 - \binom{a}{1}X^2 + \binom{a}{2}X^4 - \binom{a}{3}X^6 \pm \cdots \\ &= \sum_{\substack{m\in\mathbb{N} \\ \text{is even}}} (-1)^{m/2} \binom{a}{m/2} X^m,\end{aligned}$$

so that the X^n-coefficient of this polynomial $\left(1-X^2\right)^a$ is

$$[X^n]\left(\left(1-X^2\right)^a\right) = \begin{cases} (-1)^{n/2}\binom{a}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.}\end{cases} \tag{6}$$

On the other hand, what is the polynomial whose X^n-coefficient equals the sum

$$\sum_{k=0}^{n} (-1)^k \binom{a}{k}\binom{a}{n-k} \ ?$$

The shape of this sum is a dead giveaway; other than the letter used for the summation index (which is k, not i here), this is precisely the sum on the right hand side of (2) obtained when $P = (1-X)^a$ and $Q = (1+X)^a$. Indeed, if we define the two polynomials

$$P := (1-X)^a = \sum_{m\in\mathbb{N}} (-1)^m \binom{a}{m} X^m \qquad \text{(by Lemma 6.1.4 \textbf{(b)})}$$

and

$$Q := (1+X)^a = \sum_{m\in\mathbb{N}} \binom{a}{m} X^m \qquad \text{(by Lemma 6.1.4 \textbf{(a)})},$$

then (2) (applied to $m = n$) yields

$$\begin{aligned}[X^n](PQ) &= \sum_{i=0}^{n} (-1)^i \binom{a}{i}\cdot\binom{a}{n-i} = \sum_{i=0}^{n} (-1)^i \binom{a}{i}\binom{a}{n-i} \\ &= \sum_{k=0}^{n} (-1)^k \binom{a}{k}\binom{a}{n-k}. \end{aligned} \tag{7}$$

However, these polynomials P and Q satisfy

$$PQ = (1-X)^a (1+X)^a = \left(\underbrace{(1-X)(1+X)}_{=1-X^2}\right)^a = \left(1-X^2\right)^a.$$

Thus, we can rewrite (7) as

$$[X^n]\left(\left(1-X^2\right)^a\right) = \sum_{k=0}^{n} (-1)^k \binom{a}{k}\binom{a}{n-k}.$$

Comparing this with (6), we obtain

$$\sum_{k=0}^{n} (-1)^k \binom{a}{k}\binom{a}{n-k} = \begin{cases} (-1)^{n/2}\binom{a}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

which is precisely what we needed to prove. Thus, we have established Theorem 6.3.1. □

A more conventional proof of Theorem 6.3.1 can be found in [Grinbe15, §7.30.2].

The following neat identity is a simple consequence of Theorem 6.3.1:

Corollary 6.3.2. Let $n \in \mathbb{N}$. Then,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} (-1)^{n/2}\binom{n}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Apply Theorem 6.3.1 to $x = n$, and simplify the left hand side by noticing that

$$\binom{n}{k}\underbrace{\binom{n}{n-k}}_{=\binom{n}{k}} = \binom{n}{k}\binom{n}{k} = \binom{n}{k}^2.$$

(by Lecture 6, Theorem 1.3.9)

□

6.4. Dixon's identity

Our next goal is to show a deeper binomial identity:

Theorem 6.4.1 (Dixon's identity)**.** Let $a, b, c \in \mathbb{N}$. Then,

$$\sum_{k\in\mathbb{Z}} (-1)^k \binom{b+c}{c+k}\binom{c+a}{a+k}\binom{a+b}{b+k} = \frac{(a+b+c)!}{a!b!c!}. \tag{8}$$

Two comments are worth making here:

- It is easy to show that the sum on the LHS of (8) is well-defined (i.e., only finitely many addends are nonzero). In fact, for example, the $\binom{b+c}{c+k}$ factor is 0 whenever $k \notin \{-c, -c+1, \ldots, b\}$.
- The RHS of (8) is the multinomial coefficient $\binom{a+b+c}{a,b,c}$. (See Lecture 22, §2.10 for the definition of multinomial coefficients.)

How can we prove Theorem 6.4.1? There are various proofs (see, e.g., [Ward91]), but none of them is easy. Let me show a mysterious proof using polynomials.[4]

First, for the sake of brevity, let us denote the LHS of (8) by $F(a,b,c)$. That is, we set

$$F(a,b,c) := \sum_{k \in \mathbb{Z}} (-1)^k \binom{b+c}{c+k} \binom{c+a}{a+k} \binom{a+b}{b+k} \tag{9}$$

for all $a, b, c \in \mathbb{N}$. Thus, Dixon's identity (8) can be rewritten as

$$F(a,b,c) = \frac{(a+b+c)!}{a!b!c!}. \tag{10}$$

Our strategy for proving this identity will be to show that both of its sides satisfy the same recurrence and have the same base values. The base values are an easy matter:

Lemma 6.4.2. Dixon's identity (10) is true if $a = 0$ or $b = 0$ or $c = 0$.

Proof. WLOG assume that $a = 0$ (since the cases $b = 0$ and $c = 0$ are analogous).

[4] This proof is a particular case of I. J. Good's beautiful proof of the **Dyson conjecture** (which is a generalization of Theorem 6.4.1, although not in a very obvious way). See Good's short note [Good70] or Andrews's more detailed writeup [MacMah78, §3.3] for the latter proof.

Then,

$$\begin{aligned}
F(a,b,c) &= \sum_{k\in\mathbb{Z}} (-1)^k \binom{b+c}{c+k}\binom{c+a}{a+k}\binom{a+b}{b+k} \\
&= \sum_{k\in\mathbb{Z}} (-1)^k \binom{b+c}{c+k}\binom{c}{k}\binom{b}{b+k} \qquad (\text{since } a=0) \\
&= \sum_{\substack{k\in\mathbb{Z};\\ k<0}} (-1)^k \binom{b+c}{c+k} \underbrace{\binom{c}{k}}_{\substack{=0\\ (\text{since } k<0)}} \binom{b}{b+k} \\
&\qquad + \underbrace{(-1)^0}_{=1} \underbrace{\binom{b+c}{c+0}}_{=\binom{b+c}{c}} \underbrace{\binom{c}{0}}_{=1} \underbrace{\binom{b}{b+0}}_{=\binom{b}{b}=1} \\
&\qquad + \sum_{\substack{k\in\mathbb{Z};\\ k>0}} (-1)^k \binom{b+c}{c+k}\binom{c}{k} \underbrace{\binom{b}{b+k}}_{\substack{=0\\ (\text{since } k>0\\ \text{and thus } b+k>b)}} \\
&= \binom{b+c}{c} = \frac{(b+c)!}{b!c!} \qquad \left(\begin{array}{c}\text{an easy consequence of}\\ \text{the factorial formula for BCs}\end{array}\right) \\
&= \frac{(a+b+c)!}{a!b!c!} \qquad (\text{an easy consequence of } a=0).
\end{aligned}$$

In other words, (10) is true. This proves Lemma 6.4.2. □

Next, let us find a recurrence relation that both sides of (10) should satisfy. For the RHS, this is rather easy:

Lemma 6.4.3. Let a, b, c be positive integers. Then,

$$\frac{(a+b+c)!}{a!b!c!} = \frac{((a-1)+b+c)!}{(a-1)!b!c!} + \frac{(a+(b-1)+c)!}{a!\,(b-1)!c!} + \frac{(a+b+(c-1))!}{a!b!\,(c-1)!}.$$

Proof. Upon bringing all fractions to a common denominator (using the identity $n! = (n-1)!\cdot n$, which holds for every $n > 0$), this simplifies to $a+b+c = a+b+c$, which is indeed true.

(Alternatively, Lemma 6.4.3 is a particular case of Proposition 2.10.4 **(d)** in Lecture 22.) □

Now to the hard part: We must show that the LHS $F(a,b,c)$ of (10) satisfies the same recurrence as the RHS. In other words, we must prove that

$$F(a,b,c) = F(a-1,b,c) + F(a,b-1,c) + F(a,b,c-1)$$

for every three positive integers a, b, c.

Now is the time to pull the polynomial rabbit out of our hat. This time, we will use polynomials in **three** indeterminates X, Y, Z. We shall use the notation $\left[X^i Y^j Z^k\right] P$ for the $X^i Y^j Z^k$-coefficient of a polynomial P. For example, $\left[X^1 Y^0 Z^1\right]((2-X)(2-Y)(2-Z)) = 2$. For another example, for any $a, b, c \in \mathbb{N}$, we have

$$\left[X^a Y^b Z^c\right]\left((X+Y+Z)^{a+b+c}\right) = \binom{a+b+c}{a,b,c}$$

by the multinomial formula (Lecture 23, Theorem 2.10.5).

For any $a, b, c \in \mathbb{N}$, we define a polynomial

$$P_{a,b,c} := (Y-Z)^{b+c} \cdot (Z-X)^{c+a} \cdot (X-Y)^{a+b}. \tag{11}$$

This is a polynomial in X, Y, Z (so we might call it $P_{a,b,c}(X, Y, Z)$, but we would soon tire of carrying the X, Y, Z arguments around). The main reason why this polynomial is useful is that $F(a, b, c)$ turns out to be one of its coefficients (up to sign):

Lemma 6.4.4. For any $a, b, c \in \mathbb{N}$, we have

$$F(a, b, c) = (-1)^{a+b+c} \cdot \left[X^{2a} Y^{2b} Z^{2c}\right](P_{a,b,c}).$$

Proof. Let us first observe the following general fact: If $n, m \in \mathbb{N}$ and $k \in \mathbb{Z}$ are arbitrary, then

$$\binom{n+m}{n-k} = \binom{n+m}{m+k}. \tag{12}$$

(Indeed, this follows easily from the symmetry of BCs (Lecture 6, Theorem 1.3.9)[5].)

As we already said, the binomial formula (4) holds not just for numbers, but also for polynomials. This includes multivariate polynomials (for the same reasons). Thus, in particular, we can apply (4) to $n = b+c$, $x = Y$ and $y = -Z$. We thus obtain

$$\begin{aligned}(Y-Z)^{b+c} &= \sum_{k=0}^{b+c} \binom{b+c}{k} Y^k (-Z)^{b+c-k} = \sum_{i=0}^{b+c} \binom{b+c}{i} Y^i (-Z)^{b+c-i} \\ &\overset{0}{=} \sum_{i \in \mathbb{Z}} \binom{b+c}{i} Y^i (-Z)^{b+c-i} \end{aligned} \tag{13}$$

[5] In more detail: Let $n, m \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then, the symmetry of BCs (Lecture 6, Theorem 1.3.9) yields

$$\binom{n+m}{n-k} = \binom{n+m}{(n+m)-(n-k)} = \binom{n+m}{m+k}.$$

This proves (12).

(since every $i \in \mathbb{Z} \setminus \{0,1,\ldots,b+c\}$ satisfies $\binom{b+c}{i} = 0$). [6]

Similarly,

$$(Z-X)^{c+a} = \sum_{j\in\mathbb{Z}} \binom{c+a}{j} Z^j (-X)^{c+a-j} \tag{14}$$

and

$$(X-Y)^{a+b} = \sum_{k\in\mathbb{Z}} \binom{a+b}{k} X^k (-Y)^{a+b-k}. \tag{15}$$

Now, the definition of $P_{a,b,c}$ yields

$$\begin{aligned}
P_{a,b,c} &= (Y-Z)^{b+c} \cdot (Z-X)^{c+a} \cdot (X-Y)^{a+b} \\
&= \left(\sum_{i\in\mathbb{Z}} \binom{b+c}{i} Y^i (-Z)^{b+c-i} \right) \cdot \left(\sum_{j\in\mathbb{Z}} \binom{c+a}{j} Z^j (-X)^{c+a-j} \right) \\
&\qquad \cdot \left(\sum_{k\in\mathbb{Z}} \binom{a+b}{k} X^k (-Y)^{a+b-k} \right) \qquad \text{(by (13), (14) and (15))} \\
&= \underbrace{\sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}}}_{= \sum_{(i,j,k)\in\mathbb{Z}^3}} \\
&\qquad \underbrace{\binom{b+c}{i} Y^i (-Z)^{b+c-i} \binom{c+a}{j} Z^j (-X)^{c+a-j} \binom{a+b}{k} X^k (-Y)^{a+b-k}}_{= (-1)^{(b+c-i)+(c+a-j)+(a+b-k)} \binom{b+c}{i}\binom{c+a}{j}\binom{a+b}{k} X^{c+a-j+k} Y^{a+b-k+i} Z^{b+c-i+j}} \\
&= \sum_{(i,j,k)\in\mathbb{Z}^3} \underbrace{(-1)^{(b+c-i)+(c+a-j)+(a+b-k)}}_{\substack{= (-1)^{i+j+k} \\ \text{(since } (b+c-i)+(c+a-j)+(a+b-k) \\ = 2a+2b+2c-(i+j+k) \equiv i+j+k \bmod 2\text{)}}} \binom{b+c}{i}\binom{c+a}{j}\binom{a+b}{k} \\
&\qquad X^{c+a-j+k} Y^{a+b-k+i} Z^{b+c-i+j} \\
&= \sum_{(i,j,k)\in\mathbb{Z}^3} (-1)^{i+j+k} \binom{b+c}{i}\binom{c+a}{j}\binom{a+b}{k} X^{c+a-j+k} Y^{a+b-k+i} Z^{b+c-i+j}.
\end{aligned} \tag{16}$$

What is the coefficient of $X^{2a}Y^{2b}Z^{2c}$ in this polynomial? This coefficient is

[6] You may wonder about the negative powers of Y that are contained in this sum (in the addends with $i < 0$). However, these powers can be ignored, since they are getting multiplied with 0 anyway (since $\binom{b+c}{i} = 0$). The same holds for the negative powers of $-Z$ (which are contained in the addends with $i > b+c$).

clearly

$$\sum_{\substack{(i,j,k)\in\mathbb{Z}^3;\\ c+a-j+k=2a;\\ a+b-k+i=2b;\\ b+c-i+j=2c}} (-1)^{i+j+k} \binom{b+c}{i}\binom{c+a}{j}\binom{a+b}{k},$$

since the addend for any given triple $(i,j,k) \in \mathbb{Z}^3$ contributes to this coefficient if and only if the exponents $c+a-j+k$, $a+b-k+i$ and $b+c-i+j$ equal $2a$, $2b$ and $2c$, respectively. In other words, we have

$$\begin{aligned}&\left[X^{2a}Y^{2b}Z^{2c}\right](P_{a,b,c})\\ &= \sum_{\substack{(i,j,k)\in\mathbb{Z}^3;\\ c+a-j+k=2a;\\ a+b-k+i=2b;\\ b+c-i+j=2c}} (-1)^{i+j+k} \binom{b+c}{i}\binom{c+a}{j}\binom{a+b}{k}. \end{aligned} \tag{17}$$

Let us now simplify the sum on the RHS of this equality. The three equations $c+a-j+k = 2a$, $a+b-k+i = 2b$ and $b+c-i+j = 2c$ are dependent: More precisely, the third of them follows from the first two (since the sum of all three equations is $2a+2b+2c = 2a+2b+2c$). Thus, we can remove the third equation, transforming the sum into

$$\sum_{\substack{(i,j,k)\in\mathbb{Z}^3;\\ c+a-j+k=2a;\\ a+b-k+i=2b}} (-1)^{i+j+k} \binom{b+c}{i}\binom{c+a}{j}\binom{a+b}{k}.$$

The remaining two equations $c+a-j+k = 2a$ and $a+b-k+i = 2b$ uniquely determine i and j in terms of k: namely,

$$\begin{aligned} i &= 2b-(a+b-k) = b-a+k \qquad \text{and} \\ j &= (c+a+k)-2a = c-a+k. \end{aligned}$$

Thus, instead of summing over all triples $(i,j,k) \in \mathbb{Z}^3$ satisfying these two equations, we can just as well sum over all $k \in \mathbb{Z}$ and substitute $b-a+k$ for i

and $c-a+k$ for j in the sum. Hence,

$$\sum_{\substack{(i,j,k)\in\mathbb{Z}^3;\\ c+a-j+k=2a;\\ a+b-k+i=2b}} (-1)^{i+j+k} \binom{b+c}{i}\binom{c+a}{j}\binom{a+b}{k}$$

$$= \sum_{k\in\mathbb{Z}} (-1)^{(b-a+k)+(c-a+k)+k} \binom{b+c}{b-a+k}\binom{c+a}{c-a+k}\binom{a+b}{k}$$

$$= \sum_{k\in\mathbb{Z}} \underbrace{(-1)^{(b-a+(a-k))+(c-a+(a-k))+(a-k)}}_{\substack{=(-1)^{a+b+c-3k}=(-1)^{a+b+c+k}\\ \text{(since } -3k\equiv k \bmod 2\text{)}}} \underbrace{\binom{b+c}{b-a+(a-k)}}_{=\binom{b+c}{b-k}} \underbrace{\binom{c+a}{c-a+(a-k)}}_{=\binom{c+a}{c-k}} \binom{a+b}{a-k}$$

(here, we have substituted $a-k$ for k in the sum)

$$= \sum_{k\in\mathbb{Z}} \underbrace{(-1)^{a+b+c+k}}_{=(-1)^{a+b+c}(-1)^k} \underbrace{\binom{b+c}{b-k}}_{\substack{=\binom{b+c}{c+k}\\ \text{(by (12))}}} \underbrace{\binom{c+a}{c-k}}_{\substack{=\binom{c+a}{a+k}\\ \text{(by (12))}}} \underbrace{\binom{a+b}{a-k}}_{\substack{=\binom{a+b}{b+k}\\ \text{(by (12))}}}$$

$$= (-1)^{a+b+c} \underbrace{\sum_{k\in\mathbb{Z}} (-1)^k \binom{b+c}{c+k}\binom{c+a}{a+k}\binom{a+b}{b+k}}_{\substack{=F(a,b,c)\\ \text{(by (9))}}}$$

$$= (-1)^{a+b+c} F(a,b,c).$$

Now, (17) becomes

$$\left[X^{2a}Y^{2b}Z^{2c}\right](P_{a,b,c})$$

$$= \sum_{\substack{(i,j,k)\in\mathbb{Z}^3;\\ c+a-j+k=2a;\\ a+b-k+i=2b;\\ b+c-i+j=2c}} (-1)^{i+j+k} \binom{b+c}{i}\binom{c+a}{j}\binom{a+b}{k}$$

$$= \sum_{\substack{(i,j,k)\in\mathbb{Z}^3;\\ c+a-j+k=2a;\\ a+b-k+i=2b}} (-1)^{i+j+k} \binom{b+c}{i}\binom{c+a}{j}\binom{a+b}{k}$$

$$\left(\begin{array}{c}\text{since, as we said, the third equation } b+c-i+j=2c\\ \text{under the summation sign follows from the other two}\end{array}\right)$$

$$= (-1)^{a+b+c} F(a,b,c).$$

Solving this for $F(a,b,c)$, we obtain

$$F(a,b,c) = \frac{\left[X^{2a}Y^{2b}Z^{2c}\right](P_{a,b,c})}{(-1)^{a+b+c}} = (-1)^{a+b+c} \cdot \left[X^{2a}Y^{2b}Z^{2c}\right](P_{a,b,c}).$$

This proves Lemma 6.4.4. □

The next lemma is a straightforward computation:

Lemma 6.4.5. Let Q be a polynomial in X, Y, Z that is divisible by $(Y-Z)(Z-X)(X-Y)$ (that is, Q can be written as $(Y-Z)(Z-X)(X-Y)R$ for some polynomial R). Then,

$$\frac{X^2Q}{(X-Y)(Z-X)} + \frac{Y^2Q}{(Y-Z)(X-Y)} + \frac{Z^2Q}{(Z-X)(Y-Z)} = -Q.$$

It would be easier to restate this lemma in the form

$$\frac{X^2}{(X-Y)(Z-X)} + \frac{Y^2}{(Y-Z)(X-Y)} + \frac{Z^2}{(Z-X)(Y-Z)} = -1.$$

However, this requires working with **rational functions** (i.e., formal ratios of polynomials) instead of polynomials. To keep our arguments maximally elementary, we prefer to stay among the polynomials, and thus we only divide polynomials by other polynomials if the former are divisible by the latter.

Proof of Lemma 6.4.5. Bringing the fractions to a common denominator, we find

$$\begin{aligned}
&\frac{X^2Q}{(X-Y)(Z-X)} + \frac{Y^2Q}{(Y-Z)(X-Y)} + \frac{Z^2Q}{(Z-X)(Y-Z)} \\
&= \frac{X^2(Y-Z)Q + Y^2(Z-X)Q + Z^2(X-Y)Q}{(Y-Z)(Z-X)(X-Y)} \\
&= \frac{\left(X^2(Y-Z) + Y^2(Z-X) + Z^2(X-Y)\right)Q}{(Y-Z)(Z-X)(X-Y)} \\
&= \frac{-(Y-Z)(Z-X)(X-Y)Q}{(Y-Z)(Z-X)(X-Y)} \\
&\qquad \left(\begin{array}{c}\text{since a straightforward expansion of both sides} \\ \text{verifies that } X^2(Y-Z) + Y^2(Z-X) + Z^2(X-Y) \\ = -(Y-Z)(Z-X)(X-Y)\end{array}\right) \\
&= -Q.
\end{aligned}$$

□

The next lemma is about how the coefficients of a polynomial get shifted when that polynomial is multiplied by a power of X:

Lemma 6.4.6. Let P be a polynomial in X, Y, Z. Let $u, i, j, k \in \mathbb{N}$. Then,

$$\left[X^i Y^j Z^k\right] P = \left[X^{i+u} Y^j Z^k\right] \left(X^u P\right).$$

Proof. Let us first prove the analogous fact about polynomials in one variable: If P is a polynomial in a single variable X, and if $u, i \in \mathbb{N}$, then

$$\left[X^i\right] P = \left[X^{i+u}\right] \left(X^u P\right). \tag{18}$$

Indeed, write P in the form $P = \sum_{n \in \mathbb{N}} a_n X^n$. Then, $\left[X^i\right] P = a_i$. However, from $P = \sum_{n \in \mathbb{N}} a_n X^n$, we obtain

$$X^u P = X^u \sum_{n \in \mathbb{N}} a_n X^n = \sum_{n \in \mathbb{N}} a_n \underbrace{X^u X^n}_{= X^{u+n} = X^{n+u}} = \sum_{n \in \mathbb{N}} a_n X^{n+u},$$

which is clearly a polynomial whose X^{i+u}-coefficient is a_i. In other words, $\left[X^{i+u}\right] \left(X^u P\right) = a_i$. Comparing this with $\left[X^i\right] P = a_i$, we obtain $\left[X^i\right] P = \left[X^{i+u}\right] \left(X^u P\right)$. Thus, (18) is proved.

The case of three variables X, Y, Z is analogous; the only difference is that there are now two further variables Y and Z that passively tag along. □

Now to the crucial step:

Lemma 6.4.7. Let a, b, c be positive integers. Then,

$$F(a, b, c) = F(a-1, b, c) + F(a, b-1, c) + F(a, b, c-1).$$

Proof. Since a, b, c are positive integers, we have $a, b, c \geq 1$. Thus, the exponents $b+c$, $c+a$ and $a+b$ in (11) are ≥ 2 each. Hence, they are ≥ 1 each. Therefore, the polynomial $P_{a,b,c}$ is divisible by $(Y-Z)(Z-X)(X-Y)$.

The definition of $P_{a-1,b,c}$ yields

$$\begin{aligned}
P_{a-1,b,c} &= (Y-Z)^{b+c} \cdot \underbrace{(Z-X)^{c+a-1}}_{= \frac{(Z-X)^{c+a}}{Z-X}} \cdot \underbrace{(X-Y)^{a-1+b}}_{= \frac{(X-Y)^{a+b}}{X-Y}} \\
&= (Y-Z)^{b+c} \cdot \frac{(Z-X)^{c+a}}{Z-X} \cdot \frac{(X-Y)^{a+b}}{X-Y} \\
&= \frac{(Y-Z)^{b+c} \cdot (Z-X)^{c+a} \cdot (X-Y)^{a+b}}{(X-Y)(Z-X)} \\
&= \frac{P_{a,b,c}}{(X-Y)(Z-X)} \qquad \text{(by (11))}.
\end{aligned} \tag{19}$$

Lemma 6.4.4 yields

$$F(a,b,c) = (-1)^{a+b+c} \cdot \left[X^{2a}Y^{2b}Z^{2c}\right](P_{a,b,c}).$$

However, we can also apply Lemma 6.4.4 to $a-1$ instead of a (since $a \geq 1$, so that $a-1 \in \mathbb{N}$). Thus, we obtain

$$\begin{aligned}
F(a-1,b,c) &= \underbrace{(-1)^{a-1+b+c}}_{=-(-1)^{a+b+c}} \cdot \underbrace{\left[X^{2(a-1)}Y^{2b}Z^{2c}\right](P_{a-1,b,c})}_{=\left[X^{2(a-1)+2}Y^{2b}Z^{2c}\right]\left(X^2 P_{a-1,b,c}\right)} \\
&\qquad \text{(by Lemma 6.4.6, applied to } P=P_{a-1,b,c} \\
&\qquad \text{and } i=2(a-1) \text{ and } j=2b \text{ and } k=2c \text{ and } u=2) \\
&= -(-1)^{a+b+c} \cdot \left[\underbrace{X^{2(a-1)+2}}_{=X^{2a}} Y^{2b}Z^{2c}\right] \left(X^2 \cdot \underbrace{\underbrace{P_{a-1,b,c}}_{=\frac{P_{a,b,c}}{(X-Y)(Z-X)}}}_{\text{(by (19))}}\right) \\
&= -(-1)^{a+b+c} \cdot \left[X^{2a}Y^{2b}Z^{2c}\right] \left(X^2 \cdot \frac{P_{a,b,c}}{(X-Y)(Z-X)}\right) \\
&= -(-1)^{a+b+c} \cdot \left[X^{2a}Y^{2b}Z^{2c}\right] \frac{X^2 P_{a,b,c}}{(X-Y)(Z-X)}.
\end{aligned}$$

Similarly,

$$F(a,b-1,c) = -(-1)^{a+b+c} \cdot \left[X^{2a}Y^{2b}Z^{2c}\right] \frac{Y^2 P_{a,b,c}}{(Y-Z)(X-Y)} \qquad \text{and}$$

$$F(a,b,c-1) = -(-1)^{a+b+c} \cdot \left[X^{2a}Y^{2b}Z^{2c}\right] \frac{Z^2 P_{a,b,c}}{(Z-X)(Y-Z)}.$$

Summing these three equations together, we find

$$
\begin{aligned}
&F(a-1,b,c)+F(a,b-1,c)+F(a,b,c-1)\\
&=-(-1)^{a+b+c}\cdot\left[X^{2a}Y^{2b}Z^{2c}\right]\frac{X^2P_{a,b,c}}{(X-Y)(Z-X)}\\
&\qquad-(-1)^{a+b+c}\cdot\left[X^{2a}Y^{2b}Z^{2c}\right]\frac{Y^2P_{a,b,c}}{(Y-Z)(X-Y)}\\
&\qquad-(-1)^{a+b+c}\cdot\left[X^{2a}Y^{2b}Z^{2c}\right]\frac{Z^2P_{a,b,c}}{(Z-X)(Y-Z)}\\
&=-(-1)^{a+b+c}\cdot\left[X^{2a}Y^{2b}Z^{2c}\right]\underbrace{\left(\frac{X^2P_{a,b,c}}{(X-Y)(Z-X)}+\frac{Y^2P_{a,b,c}}{(Y-Z)(X-Y)}+\frac{Z^2P_{a,b,c}}{(Z-X)(Y-Z)}\right)}_{=-P_{a,b,c}\text{ (by Lemma 6.4.5, applied to } Q=P_{a,b,c})}\\
&\qquad\left(\begin{array}{c}\text{since } \left(\left[X^iY^jZ^k\right]U\right)+\left(\left[X^iY^jZ^k\right]V\right)=\left[X^iY^jZ^k\right](U+V)\\ \text{for any } i,j,k\in\mathbb{N} \text{ and any two polynomials } U,V\end{array}\right)\\
&=-(-1)^{a+b+c}\cdot\underbrace{\left[X^{2a}Y^{2b}Z^{2c}\right](-P_{a,b,c})}_{=-\left[X^{2a}Y^{2b}Z^{2c}\right]P_{a,b,c}}\\
&=(-1)^{a+b+c}\cdot\left[X^{2a}Y^{2b}Z^{2c}\right](P_{a,b,c})=F(a,b,c)\qquad\text{(by Lemma 6.4.4)}.
\end{aligned}
$$

This proves Lemma 6.4.7. □

We are now ready to prove Theorem 6.4.1:

Proof of Theorem 6.4.1. We must prove (8). As we recall, the equality (10) is just a restatement of (8), so it suffices to prove (10) instead.

The proof is completely straightforward at this point: The LHS $F(a,b,c)$ of (10) satisfies a recurrence (Lemma 6.4.7), but the RHS $\dfrac{(a+b+c)!}{a!b!c!}$ satisfies the same recurrence (by Lemma 6.4.3). Furthermore, the starting values (i.e., the values when one of a,b,c is 0) are also the same (by Lemma 6.4.2). It thus follows that the LHS and the RHS are always equal, so that (10) holds.

For the sake of completeness, here is the proof in more detail:

We shall prove (10) by induction on $a+b+c$.

Base case: If $a+b+c=0$, then $a=b=c=0$, and thus (10) holds by Lemma 6.4.2.

Induction step: Let m be a positive integer. Assume (as the induction hypothesis) that (10) holds whenever $a+b+c=m-1$. We must now prove (10) in the case when $a+b+c=m$.

So let $a,b,c\in\mathbb{N}$ satisfy $a+b+c=m$. We must prove (10). If $a=0$ or $b=0$ or $c=0$, then (10) follows directly from Lemma 6.4.2. Thus, we WLOG assume that none of a,b,c is 0. Hence, a,b,c are positive integers. Thus, Lemma 6.4.7 yields

$$F(a,b,c)=F(a-1,b,c)+F(a,b-1,c)+F(a,b,c-1). \tag{20}$$

However, from $a+b+c=m$, we obtain $(a-1)+b+c=\underbrace{a+b+c}_{=m}-1=m-1$. Thus, by our induction hypothesis, (10) holds for $a-1$ instead of a. In other words, we have

$$F(a-1,b,c)=\frac{((a-1)+b+c)!}{(a-1)!b!c!}.$$

Similarly,

$$F(a,b-1,c)=\frac{(a+(b-1)+c)!}{a!\,(b-1)!c!} \qquad \text{and}$$
$$F(a,b,c-1)=\frac{(a+b+(c-1))!}{a!b!\,(c-1)!}.$$

In light of these three equalities, we can rewrite (20) as

$$\begin{aligned} F(a,b,c) &= \frac{((a-1)+b+c)!}{(a-1)!b!c!}+\frac{(a+(b-1)+c)!}{a!\,(b-1)!c!}+\frac{(a+b+(c-1))!}{a!b!\,(c-1)!} \\ &= \frac{(a+b+c)!}{a!b!c!} \qquad \text{(by Lemma 6.4.3)}. \end{aligned}$$

In other words, (10) holds for our a,b,c. This completes the induction step. Thus, by induction, we have proved (10), and with it Theorem 6.4.1. □

Oof. I don't remember where I learned the above proof, nor do I have a good explanation of "where it comes from" (or, more pragmatically, how to find such proofs). However, it is an impressive exhibition of the power of polynomials.

Let us mention the following consequence of Theorem 6.4.1 (similar to Corollary 6.3.2, but significantly harder):

Corollary 6.4.8. Let $n\in\mathbb{N}$. Then,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=\begin{cases}(-1)^{n/2}\dfrac{(3n/2)!}{(n/2)!^3}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd.}\end{cases}$$

Proof sketch. When n is odd, the sum $\sum\limits_{k=0}^{n}(-1)^k\binom{n}{k}^3$ is easily seen to be 0 since its addends cancel out in pairs (namely, $(-1)^k\binom{n}{k}^3$ cancels $(-1)^{n-k}\binom{n}{n-k}^3$). So we

WLOG assume that n is even. Then,

$$\begin{aligned}
&\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 \\
&\overset{0}{=} \sum_{k\in\mathbb{Z}} (-1)^k \binom{n}{k}^3 \\
&= \sum_{k\in\mathbb{Z}} \underbrace{(-1)^{n/2+k}}_{=(-1)^{n/2}(-1)^k} \binom{n}{n/2+k}^3 \qquad \left(\begin{array}{c}\text{here, we have substituted } n/2+k \\ \text{for } k \text{ in the sum}\end{array}\right) \\
&= (-1)^{n/2} \sum_{k\in\mathbb{Z}} (-1)^k \underbrace{\binom{n}{n/2+k}^3}_{=\binom{n/2+n/2}{n/2+k}^3} \\
&\qquad\qquad = \underbrace{\binom{n/2+n/2}{n/2+k}\binom{n/2+n/2}{n/2+k}\binom{n/2+n/2}{n/2+k}}_{=\frac{(n/2+n/2+n/2)!}{(n/2)!\,(n/2)!\,(n/2)!}} \\
&= (-1)^{n/2} \sum_{k\in\mathbb{Z}} (-1)^k \binom{n/2+n/2}{n/2+k}\binom{n/2+n/2}{n/2+k}\binom{n/2+n/2}{n/2+k} \\
&\qquad\qquad \text{(by Theorem 6.4.1, applied to } a{=}n/2 \text{ and } b{=}n/2 \text{ and } c{=}n/2) \\
&= (-1)^{n/2} \frac{(n/2+n/2+n/2)!}{(n/2)!\,(n/2)!\,(n/2)!} = (-1)^{n/2} \frac{(3n/2)!}{(n/2)!^3}.
\end{aligned}$$

Thus, Corollary 6.4.8 is proved. □

With Corollary 6.3.2 and Corollary 6.4.8, we have paid off all our proof debts from Lecture 15, Remark 2.3.5. As we said, neither $\sum_{k=0}^{n} \binom{n}{k}^3$ nor $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^4$ seem to have explicit expressions, so the series of binomial identities we have proved has come to a natural end.

And so has this lecture. More applications of generating functions can be found in [21s, Chapter 3], [Loehr11, Chapter 7 (in the 1st edition)], [Sambal22], [Wilf04], [Stanle11] and many other texts on combinatorics (as well as probability theory, where they are perhaps even more central).

References

[21s] Darij Grinberg, *Algebraic Combinatorics (Drexel Spring 2021 Math 701 lecture notes)*, 4 May 2021.
`https://www.cip.ifi.lmu.de/~grinberg/t/21s/lecs.pdf`

[Good70] I. J. Good, *Short Proof of a Conjecture by Dyson*, Journal of Mathematical Physics **11**, no. 6, 1970, p. 1884.

[Grinbe15] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 15 September 2022.
http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2022-09-15c .

[Loehr11] Nicholas A. Loehr, *Bijective Combinatorics*, Chapman & Hall/CRC 2011.

[MacMah78] Percy Alexander MacMahon, *Collected Papers: Volume I (Combinatorics)*, edited by George E. Andrews, MIT Press 1978.

[Niven69] Ivan Niven, *Formal Power Series*, The American Mathematical Monthly **76**, No. 8 (Oct., 1969), pp. 871–889.
https://www.maa.org/programs/maa-awards/writing-awards/formal-power-series

[Sambal22] Benjamin Sambale, *An invitation to formal power series*, arXiv:2205.00879v2.

[Stanle11] Richard P. Stanley, *Enumerative Combinatorics, volume 1*, Second edition, version of 15 July 2011. Available at http://math.mit.edu/~rstan/ec/ .
See http://math.mit.edu/~rstan/ec/ for errata.

[Ward91] James Ward, *100 Years of Dixon's Identity*, IMS Bulletin **27** (1991), pp. 46–54.

[Wilf04] Herbert S. Wilf, *generatingfunctionology*, 2nd edition 2004.
https://www.math.upenn.edu/~wilf/DownldGF.html