Math 222 Fall 2022, Lecture 28: Permutations

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

4. Permutations

4.3. The cycle decomposition of a permutation (cont'd)

4.3.5. Cycles and transpositions

Recall that if α and β are two permutations of a finite set X, then we denote their composition $\alpha \circ \beta$ by $\alpha\beta$. More generally, the **product** of any number of permutations $\alpha_1, \alpha_2, \ldots, \alpha_k$ is defined to be their composition $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$, and will be denoted by $\alpha_1\alpha_2 \cdots \alpha_k$. When $k = 0$, this composition is a composition of 0 maps, and this composition is defined to be the identity map id_X. (A composition of 0 maps is always defined to be the identity map.)

Recall that a transposition of a set X means a permutation of the form $t_{i,j}$, where i and j are two distinct elements of X. This permutation $t_{i,j}$ swaps i with j and leaves all remaining elements of X unchanged.

It turns out that every permutation of a finite set can be written as a product of transpositions (and this is a useful fact, since transpositions are easier to understand than permutations in general). We can even be more specific:

Theorem 4.3.23. Let $n \in \mathbb{N}$. Let X be an n-element set. Then:

(a) Every permutation $\sigma \in S_X$ can be written as a product of at most $n - 1$ transpositions (if $n > 0$).

(b) Let $\sigma \in S_X$ be any permutation. Let

$$k = n - (\text{\# of orbits of } \sigma) .$$

Then, σ can be written as a product of k transpositions, but not of fewer than k transpositions. In other words, k is the smallest $p \in \mathbb{N}$ such that σ can be written as a product of p transpositions.

Example 4.3.24. Let σ be the permutation of $[10]$ whose OLN is

$$5\ 4\ 3\ 2\ 6\ \ (10)\ \ 1\ 9\ 8\ 7.$$

(This is the permutation from Example 4.1.5 in Lecture 26.) As we know, σ has 4 orbits. Thus, Theorem 4.3.23 **(b)** (applied to $n = 10$ and $X = [10]$ and $k = n - 4 = 10 - 4 = 6$) shows that σ can be written as a product of 6 transpositions, but not of fewer than 6 transpositions. And indeed, we can easily verify that σ can be written as a product of 6 transpositions:

$$\sigma = t_{1,5} \circ t_{5,6} \circ t_{6,10} \circ t_{7,10} \circ t_{8,9} \circ t_{2,4}.$$

(There are also several other ways to write σ in this form.)

To prove Theorem 4.3.23, we need a little lemma, which is an easy consequence of Lemma 4.3.13 in Lecture 27:

Lemma 4.3.25. Let X be a finite set. Let i and j be two distinct elements of X. Let $\sigma \in S_X$ be a permutation, and let $\tau = t_{i,j} \circ \sigma$. Then:

(a) If $i \overset{\sigma}{\sim} j$, then (# of orbits of τ) $=$ (# of orbits of σ) $+ 1$.

(b) If we don't have $i \overset{\sigma}{\sim} j$, then (# of orbits of τ) $=$ (# of orbits of σ) $- 1$.

(c) In either case, we have (# of orbits of τ) $\geq$ (# of orbits of σ) $- 1$.

Proof of Lemma 4.3.25 (sketched). **(a)** This is precisely Lemma 4.3.13 **(a)** in Lecture 27.

(b) This is precisely Lemma 4.3.13 **(c)** in Lecture 27.

(c) We either have $i \overset{\sigma}{\sim} j$, or we don't have $i \overset{\sigma}{\sim} j$. In the former case, Lemma 4.3.25 **(a)** yields (# of orbits of τ) $=$ (# of orbits of σ) $+ 1 \geq$ (# of orbits of σ) $- 1$. In the latter case, Lemma 4.3.25 **(b)** yields (# of orbits of τ) $=$ (# of orbits of σ) $- 1$. In either case, the claim of Lemma 4.3.25 **(c)** follows. □

Proof of Theorem 4.3.23 (sketched). **(b)** We first observe that $k \geq 0$ (since the # orbits of σ cannot be larger than n), so that $k \in \mathbb{N}$. Thus, we can induct on k:

Base case: The case $k = 0$ can happen only if $\sigma = \mathrm{id}_X$ (because $k = 0$ means that σ has n orbits, whence every orbit is a 1-element set, so that each $x \in X$ is a fixed point of σ, and therefore $\sigma = \mathrm{id}_X$). In this case, however, σ can be written as a product of 0 transpositions (since a product of 0 transpositions is id_X by definition), but not of fewer than 0 transpositions (since a product cannot have fewer than 0 factors). Thus, Theorem 4.3.23 **(b)** is proved in the case $k = 0$.

Induction step: Let $h \in \mathbb{N}$. Assume (as the induction hypothesis) that Theorem 4.3.23 **(b)** is proved in the case when $k = h$. We must prove that Theorem 4.3.23 **(b)** holds in the case when $k = h + 1$.

So let σ be any permutation with

$$n - (\text{\# of orbits of } \sigma) = h + 1. \tag{1}$$

We must show that σ can be written as a product of $h + 1$ transpositions, but not of fewer than $h + 1$ transpositions.

From (1), we obtain (# of orbits of σ) $= n - \underbrace{(h+1)}_{>0} < n$, and therefore there are two elements i and j of X that belong to the same orbit of σ (why?). Consider such i and j. Thus, $i \overset{\sigma}{\sim} j$. Let τ be the permutation $t_{i,j} \circ \sigma$. Then,

$$n - \underbrace{(\text{\# of orbits of } \tau)}_{\substack{=(\text{\# of orbits of } \sigma)+1 \\ (\text{by Lemma 4.3.25 } \textbf{(a)})}} = n - ((\text{\# of orbits of } \sigma) + 1) = \underbrace{n - (\text{\# of orbits of } \sigma)}_{=h+1} - 1 = h.$$

Hence, by the induction hypothesis, the permutation τ can be written as a product of h transpositions, but not of fewer than h transpositions. In particular, τ can be written as a product of h transpositions. In other words, τ can be written as

$$\tau = t_{i_1,j_1} \circ t_{i_2,j_2} \circ \cdots \circ t_{i_h,j_h}$$

for some $i_1, j_1, i_2, j_2, \ldots, i_h, j_h \in X$. Consider these $i_1, j_1, i_2, j_2, \ldots, i_h, j_h$.

Recall that $t_{i,j} \circ t_{i,j} = \mathrm{id}_X$. Now, $t_{i,j} \circ \underbrace{\tau}_{= t_{i,j} \circ \sigma} = \underbrace{t_{i,j} \circ t_{i,j}}_{= \mathrm{id}_X} \circ \sigma = \mathrm{id}_X \circ \sigma = \sigma$. Hence,

$$\sigma = t_{i,j} \circ \underbrace{\tau}_{= t_{i_1,j_1} \circ t_{i_2,j_2} \circ \cdots \circ t_{i_h,j_h}} = t_{i,j} \circ t_{i_1,j_1} \circ t_{i_2,j_2} \circ \cdots \circ t_{i_h,j_h}.$$

This shows that σ can be written as a product of $h+1$ transpositions. It remains to prove that σ cannot be written as a product of fewer than $h+1$ transpositions.

To prove this, we assume the contrary. Thus, σ can be written as

$$\sigma = t_{u_1,v_1} \circ t_{u_2,v_2} \circ \cdots \circ t_{u_p,v_p}$$

for some nonnegative integer $p < h+1$ and some elements $u_1, v_1, u_2, v_2, \ldots, u_p, v_p \in X$. Consider this p and these $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$. For each $i \in [p+1]$, we set

$$\sigma_i := t_{u_i,v_i} \circ t_{u_{i+1},v_{i+1}} \circ \cdots \circ t_{u_p,v_p}.$$

Then, $\sigma_1 = t_{u_1,v_1} \circ t_{u_2,v_2} \circ \cdots \circ t_{u_p,v_p} = \sigma$, whereas $\sigma_{p+1} = \mathrm{id}_X$ (since $\sigma_{p+1} = t_{u_{p+1},v_{p+1}} \circ t_{u_{p+2},v_{p+2}} \circ \cdots \circ t_{u_p,v_p}$ is a composition of 0 permutations and therefore equals id_X by definition). For each $i \in [p]$, we have $\sigma_i = t_{u_i,v_i} \circ \sigma_{i+1}$ (to see why, compare the definitions of σ_i and σ_{i+1}) and therefore

$$(\#\text{ of orbits of } \sigma_i) \geq (\#\text{ of orbits of } \sigma_{i+1}) - 1 \qquad (2)$$

(by Lemma 4.3.25 **(c)**, applied to σ_{i+1}, u_i, v_i and σ_i instead of σ, i, j and τ). Hence,

$$\begin{aligned}
(\#\text{ of orbits of } \sigma) &= (\#\text{ of orbits of } \sigma_1) \qquad (\text{since } \sigma = \sigma_1) \\
&\geq \underbrace{(\#\text{ of orbits of } \sigma_2)}_{\substack{\geq (\#\text{ of orbits of } \sigma_3) - 1 \\ (\text{by (2)})}} - 1 \qquad (\text{by (2)}) \\
&\geq \underbrace{(\#\text{ of orbits of } \sigma_3)}_{\substack{\geq (\#\text{ of orbits of } \sigma_4) - 1 \\ (\text{by (2)})}} - 2 \\
&\geq \underbrace{(\#\text{ of orbits of } \sigma_4)}_{\substack{\geq (\#\text{ of orbits of } \sigma_5) - 1 \\ (\text{by (2)})}} - 3 \\
&\geq \cdots \\
&\geq \underbrace{(\#\text{ of orbits of } \sigma_{p+1})}_{\substack{= n \\ (\text{since } \sigma_{p+1} = \mathrm{id}_X \text{ has } n \text{ orbits})}} - p = n - p.
\end{aligned}$$

However, (1) yields

$$h + 1 = n - \underbrace{(\#\text{ of orbits of } \sigma)}_{\geq n - p} \leq n - (n - p) = p.$$

This contradicts $p < h+1$. This contradiction shows that our assumption was false. Hence, we have proved that σ cannot be written as a product of fewer than $h+1$ transpositions. This completes the induction step, and thus Theorem 4.3.23 **(b)** is proved.

(a) Assume that $n > 0$. Then, σ has at least 1 orbit (why?). In other words, $(\text{\# of orbits of } \sigma) \geq 1$.

Let $k = n - (\text{\# of orbits of } \sigma)$. Then, $k = n - \underbrace{(\text{\# of orbits of } \sigma)}_{\geq 1} \leq n - 1$. However, Theorem 4.3.23 **(b)** yields that σ can be written as a product of k transpositions. Hence, σ can be written as a product of at most $n-1$ transpositions (since $k \leq n-1$). This proves Theorem 4.3.23 **(a)**. □

We can replace "transpositions" by "distinct transpositions" in Theorem 4.3.23 (but we cannot force the transpositions to commute with each other; thus, the order of factors in the product matters).

For $X = [n]$, we can state something stronger:

Theorem 4.3.26. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Let $\operatorname{Nonfix} \sigma$ denote the set of all $x \in [n]$ that satisfy $\sigma(x) \neq x$ (that is, the set of all elements of $[n]$ that are not fixed points of σ). Then, we can write σ as a product of k transpositions

$$\begin{aligned} \sigma &= t_{i_1,j_1} \circ t_{i_2,j_2} \circ \cdots \circ t_{i_k,j_k} \qquad \text{with} \\ \operatorname{Nonfix} \sigma &= \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\}, \\ k &= n - (\text{\# of orbits of } \sigma), \\ i_1 &< i_2 < \cdots < i_k \qquad \text{and} \\ i_p &< j_p \text{ for each } p \in [k]. \end{aligned} \tag{3}$$

(The equality (3) does not imply that the $2k$ elements $i_1, j_1, i_2, j_2, \ldots, i_k, j_k$ are all distinct; any element of $\operatorname{Nonfix} \sigma$ can appear multiple times in this list.)

Proof idea. Perform the same induction as in the above proof of Theorem 4.3.23 **(b)** (with $X = [n]$), but make sure to choose i and j strategically in the induction step: Namely, let i be the smallest element of $\operatorname{Nonfix} \sigma$, and let $j = \sigma(i)$. Then, $\tau = t_{i,j} \circ \sigma$ has 1 more orbit than σ and satisfies $\tau(i) = i$ and $\operatorname{Nonfix} \sigma = \{i, j\} \cup \operatorname{Nonfix} \tau$ and $\operatorname{Nonfix} \tau \subseteq \{i+1, i+2, \ldots, n\}$. This allows the induction hypothesis to be used. Details are left to the reader. □

4.3.6. Permutations and partitions

Permutations are connected with set partitions and also with integer partitions. To see how, fix a finite set X. Then, for every permutation $\sigma \in S_X$, the orbits of σ form a set partition of X (since each element of X belongs to exactly one orbit of σ). Thus, we obtain a map

$$\begin{aligned} S_X = \{\text{permutations of } X\} &\to \{\text{set partitions of } X\}, \\ \sigma &\mapsto \{\text{orbits of } \sigma\}. \end{aligned}$$

For example, this map sends the permutation σ from Example 4.3.24 to the set partition $\{\{1,5,6,10,7\}, \{8,9\}, \{3\}, \{2,4\}\}$.

Furthermore, each set partition $\mathcal{F}$ of X yields an integer partition of $|X|$; the entries of the latter partition are just the sizes of all parts of $\mathcal{F}$ (listed in decreasing order). Thus, we obtain a map

$$\{\text{set partitions of } X\} \to \{\text{integer partitions of } |X|\},$$
$$\left\{ \underbrace{X_1, X_2, \ldots, X_k}_{\text{distinct}} \right\} \mapsto (|X_1|, |X_2|, \ldots, |X_k|) \text{ sorted in decreasing order.}$$

For example, this map sends the set partition $\{\{1,5,6,10,7\}, \{8,9\}, \{3\}, \{2,4\}\}$ of $[10]$ to the partition $(5,2,2,1)$.

Composing these two maps, we obtain a map

$$S_X \to \{\text{integer partitions of } |X|\},$$
$$\sigma \mapsto (\text{list of sizes of all orbits of } \sigma \text{, in decreasing order}).$$

This latter list is called the **cycle type** of σ. In other words:

Definition 4.3.27. Let X be a finite set. Let $\sigma \in S_X$ be a permutation of X. Then, the **cycle type** of σ is defined to be the partition whose parts are the sizes of the orbits of σ (written in decreasing order). This is a partition of $|X|$.

Example 4.3.28.

- A transposition $t_{i,j} \in S_X$ always has cycle type $(2,1,1,\ldots,1)$ (with $|X|-2$ many 1's).
- The identity map id_X has cycle type $(1,1,\ldots,1)$ (with $|X|$ many 1's).
- More generally, a k-cycle in S_X always has cycle type $(k,1,1,\ldots,1)$ (with $|X|-k$ many 1's after the k).
- The permutation σ from Example 4.3.24 has cycle type $(5,2,2,1)$.

The cycle type of a permutation σ determines the lengths of the cycles on the cycle digraph of σ. Thus, if you know the cycle type of a permutation $\sigma \in S_X$, then you can draw the cycle digraph of σ except for the labels of the nodes (i.e., you don't know which node corresponds to which element of X). Hence, the cycle type of a permutation determines this permutation up to "relabelling its elements". So we can view integer partitions as "unlabelled permutations".

To formalize this idea, we define the notion of **isomorphism of permutations**:

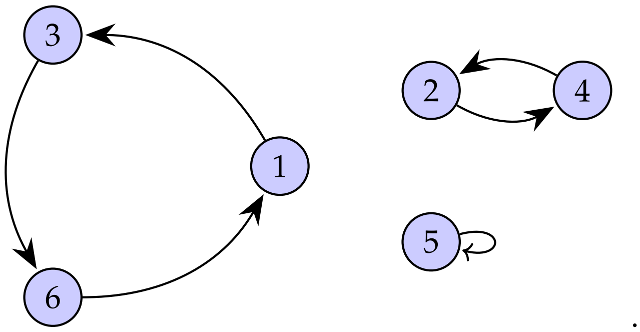
Definition 4.3.29. Let X and Y be two finite sets.

Let $\sigma \in S_X$ and $\tau \in S_Y$ be two permutations.

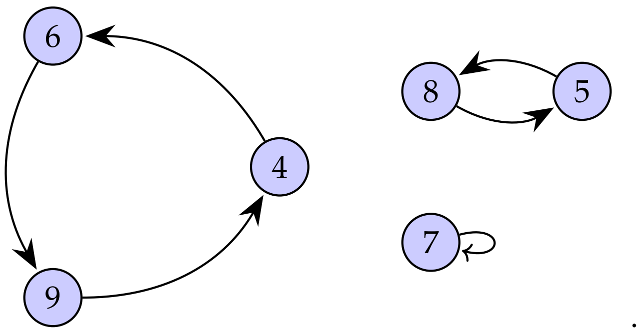
We say that these two permutations σ and τ are **isomorphic** (as permutations) if there exists a bijection $\phi : X \to Y$ such that $\tau = \phi \circ \sigma \circ \phi^{-1}$. (Recall that $\phi \circ \sigma \circ \phi^{-1}$ is the result of "relabelling" the elements of X using ϕ in the permutation σ; we have already used this construction in the proof of

Lemma 1.7.6 in Lecture 12. Thus, the equality $\tau = \phi \circ \sigma \circ \phi^{-1}$ means that if we relabel each node x in the cycle digraph of σ as $\phi(x)$, then we obtain the cycle digraph of τ.)

Example 4.3.30. Let X be the 6-element set $[6]$, and let $\sigma \in S_X$ be the permutation with cycle digraph

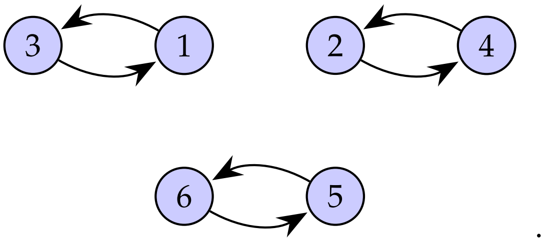


Let Y be the 6-element set $\{4, 5, 6, 7, 8, 9\}$, and let $\tau \in S_Y$ be the permutation with cycle digraph



It is obvious from the two cycle digraphs that τ is just σ with the elements $1, 2, 3, 4, 5, 6$ relabelled as $4, 8, 6, 5, 7, 9$, respectively. In other words, $\tau = \phi \circ \sigma \circ \phi^{-1}$, where $\phi : X \to Y$ is the bijection that sends $1, 2, 3, 4, 5, 6$ to $4, 8, 6, 5, 7, 9$, respectively. Thus, σ and τ are isomorphic. (Note that there are several bijections ϕ that satisfy $\tau = \phi \circ \sigma \circ \phi^{-1}$; we just picked the one most obvious from the picture. We could just as well have taken, e.g., the one that sends $1, 2, 3, 4, 5, 6$ to $6, 5, 9, 8, 7, 4$, respectively.)

On the other hand, let $\sigma' \in S_X$ be the permutation with cycle digraph



Then, σ' is not isomorphic to σ. This is easiest to see by observing that σ has a fixed point whereas σ' has none. (The property "has a fixed point" is preserved under isomorphism, since a bijection $\phi : X \to Y$ satisfying $\tau = \phi \circ \sigma \circ \phi^{-1}$ would send any fixed points of σ to fixed points of τ.)

Two isomorphic permutations of the same set X are also said to be **conjugate** in the symmetric group S_X.

Now, what we said above about cycle types can be stated rigorously as follows:

Theorem 4.3.31 (isomorphism criterion for permutations)**.** Let X and Y be two finite sets.

Let $\sigma \in S_X$ and $\tau \in S_Y$ be permutations.

Then, the permutations σ and τ are isomorphic if and only if they have the same cycle type.

Proof idea. $\Longrightarrow$: Assume that σ and τ are isomorphic. Thus, there exists a bijection $\phi : X \to Y$ satisfying $\tau = \phi \circ \sigma \circ \phi^{-1}$. This bijection ϕ must then send each orbit of σ to an orbit of τ (why?). It furthermore preserves the sizes of these orbits (since a bijection preserves the sizes of all subsets). Hence, the orbits of τ have the same sizes as the orbits of σ. In other words, σ and τ have the same cycle type.

$\Longleftarrow$: Assume that σ and τ have the same cycle type. Let $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ be this cycle type.

The permutation σ has cycle type $(\lambda_1, \lambda_2, \ldots, \lambda_k)$. In other words, it has an orbit X_1 of size λ_1, an orbit X_2 of size λ_2, an orbit X_3 of size λ_3, and so on, and these altogether k orbits $X_1, X_2, \ldots, X_k$ form a set partition of the set X. Consider these k orbits. For each $i \in [k]$, write the orbit X_i in the form $X_i = [x_i]_\sim$ for some $x_i \in X_i$, where $\sim$ is the relation $\overset{\sigma}{\sim}$. Then, for each $i \in [k]$, we have

$$X_i = \left\{\sigma^0 (x_i), \sigma^1 (x_i), \sigma^2 (x_i), \ldots, \sigma^{\lambda_i - 1} (x_i)\right\}$$

(why? hint: use Proposition 4.3.4 **(b)** in Lecture 27).

Similarly, we can analyze the orbits of τ. We learn that there are k of them, and we can denote them as $Y_1, Y_2, \ldots, Y_k$ and have

$$Y_i = \left\{\tau^0 (y_i), \tau^1 (y_i), \tau^2 (y_i), \ldots, \tau^{\lambda_i - 1} (y_i)\right\} \qquad \text{for each } i \in [k]$$

(where the elements y_i are chosen appropriately).

Note that each element of X belongs to exactly one orbit X_i of σ, and thus can be written in the form $\sigma^j (x_i)$ for a unique $i \in [k]$ and a unique $j \in \{0, 1, \ldots, \lambda_i - 1\}$. Similarly, each element of Y can be written in the form $\tau^j (y_i)$ for a unique $i \in [k]$ and a unique $j \in \{0, 1, \ldots, \lambda_i - 1\}$.

This allows us to define $\phi : X \to Y$ to be the map that sends each $\sigma^j (x_i) \in X$ (for each $i \in [k]$ and each $j \in \{0, 1, \ldots, \lambda_i - 1\}$) to the respective $\tau^j (y_i) \in Y$. This map ϕ is a bijection (why?) and satisfies $\tau \circ \phi = \phi \circ \sigma$ (why?), thus $\tau = \phi \circ \sigma \circ \phi^{-1}$. This shows that σ and τ are isomorphic. □

As a curiosity, let me mention another relation between permutations and partitions: a formula for the # of times that two permutations of a given set X commute. This formula is surprising in its simplicity:

Theorem 4.3.32. Let $n \in \mathbb{N}$ and let X be an n-element set. Then,

$$(\text{\# of pairs } (\alpha, \beta) \in S_X \times S_X \text{ such that } \alpha\beta = \beta\alpha) = n! \cdot p(n).$$

Here, $p(n)$ denotes the # of all partitions of n.

Proof idea. This theorem's natural habitat is finite group theory. In fact, if G is any finite group, then the # of pairs $(\alpha, \beta) \in G \times G$ such that $\alpha\beta = \beta\alpha$ equals $|G| \cdot \left|\widetilde{G}\right|$, where $\widetilde{G}$ is the set of all conjugacy classes of G. See `https://math.stackexchange.com/a/1401276/` or `https://math.stackexchange.com/a/3023502/` for a proof of this fact (which appears to have been first found by Erdös and Turán in 1968). Applying this fact to $G = S_X$ yields the claim of Theorem 4.3.32, after observing that the conjugacy classes of S_X are in bijection with the partitions of n (an easy consequence of Theorem 4.3.31). □

4.4. Inversions and lengths

In the previous section, we have mostly been discussing permutations of arbitrary finite sets X. Let us next discuss some features that are specific to permutations of $[n]$ (or, more generally, of totally ordered sets, but for us $[n]$ will suffice). Specifically, we will study the inversions and the length of a permutation, and find out how to express any permutation of $[n]$ as a product of simple transpositions (as opposed to arbitrary transpositions, which we know how to do from Theorem 4.3.23). We will only scratch the surface (and only sketch the proof); more details can be found in [21s, §5.3] and in [Grinbe15, §5.2, §5.5, §5.8].

4.4.1. Definitions

Definition 4.4.1. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$ be a permutation.

(a) An **inversion** of σ means a pair $(i, j) \in [n] \times [n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

(b) The **Coxeter length** $\ell(\sigma)$ of σ (also denoted $\operatorname{inv} \sigma$) is the # of inversions of σ.

Example 4.4.2. Consider the permutation $\sigma \in S_5$ whose OLN is 31452. (Thus, $\sigma(1) = 3$ and $\sigma(2) = 1$ and $\sigma(3) = 4$ and $\sigma(4) = 5$ and $\sigma(5) = 2$.)

Then, the inversions of σ are

$$\begin{aligned}
&(1,2) && (\text{since } 1 < 2 \text{ but } \sigma(1) = 3 > 1 = \sigma(2)), \\
&(1,5) && (\text{since } 1 < 5 \text{ but } \sigma(1) = 3 > 2 = \sigma(5)), \\
&(3,5) && (\text{since } 3 < 5 \text{ but } \sigma(3) = 4 > 2 = \sigma(5)), \\
&(4,5) && (\text{since } 4 < 5 \text{ but } \sigma(4) = 5 > 2 = \sigma(5)).
\end{aligned}$$

So the Coxeter length $\ell(\sigma)$ of σ is 4.

Note that the inversions of a permutation $\sigma \in S_n$ can be easily read off from its OLN: Namely, every time you see a bigger number before[1] a smaller number in the OLN of σ, you have an inversion. Note, however, that the entries of the inversion are **not** the bigger number and the smaller number, but rather their **positions** in the OLN. For example, the permutation $\sigma \in S_3$ with OLN 312 has inversion $(1,2)$ not because the 1 stands before the 2 (it doesn't), but because the entry in position 1 is larger than the entry in position 2.

Recall: Given $n \in \mathbb{N}$, we define the **simple transpositions** $s_1, s_2, \ldots, s_{n-1} \in S_n$ by $s_i := t_{i,i+1}$. Each of these simple transpositions has Coxeter length $\ell(s_i) = 1$ (since its only inversion is $(i, i+1)$, as you can easily check).

4.4.2. Counting

Now, another counting question suggests itself: How many permutations $\sigma \in S_n$ have Coxeter length k for a given n and a given k ?

Some simple results first:

Proposition 4.4.3. Let $n \in \mathbb{N}$. Then:

(a) For every $\sigma \in S_n$, we have $\ell(\sigma) \in \left\{0, 1, \ldots, \binom{n}{2}\right\}$.

(b) The only permutation $\sigma \in S_n$ satisfying $\ell(\sigma) = 0$ is $\mathrm{id}_{[n]}$.

(c) The only permutation $\sigma \in S_n$ satisfying $\ell(\sigma) = \binom{n}{2}$ is the permutation that sends each $k \in [n]$ to $n+1-k$. Its OLN is $(n, n-1, \ldots, 2, 1)$. It is occasionally denoted by w_0.

(d) Assume that $n > 0$. Then, there are exactly $n-1$ permutations $\sigma \in S_n$ satisfying $\ell(\sigma) = 1$, namely the simple transpositions $s_1, s_2, \ldots, s_{n-1}$.

Proof idea. **(a)** There are only $\binom{n}{2}$ different pairs $(i,j) \in [n] \times [n]$ with $i < j$.

(b) If a permutation $\sigma \in S_n$ satisfies $\ell(\sigma) = 0$, then it must satisfy $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(n)$ (since $\sigma(i) > \sigma(i+1)$ would cause $(i, i+1)$ to be an inversion of σ) and

[1] "Before" does not mean "immediately before". For instance, 3 is before 2 in 312.

thus $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ (since σ is injective), which easily leads to $\sigma = \mathrm{id}_{[n]}$ (why?[2]).

(c) Left to the reader (similar to part **(b)**).

(d) Left to the reader (nice exercise). □

Okay, but what about the other possible Coxeter lengths? As there is no simple explicit answer to our counting question, we introduce a notation for it:

Definition 4.4.4. Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then, $w(n,k)$ shall denote the # of permutations $\sigma \in S_n$ satisfying $\ell(\sigma) = k$.

Thus,

- we have $w(n,0) = 1$ for any $n \in \mathbb{N}$ (by Proposition 4.4.3 **(b)**);
- we have $w\left(n, \dbinom{n}{2}\right) = 1$ for any $n \in \mathbb{N}$ (by Proposition 4.4.3 **(c)**);
- we have $w(n,1) = n-1$ for any $n > 0$ (by Proposition 4.4.3 **(d)**);
- we have $w(n,k) = 0$ for every $n \in \mathbb{N}$ and $k \notin \left\{0, 1, \ldots, \dbinom{n}{2}\right\}$ (by Proposition 4.4.3 **(a)**).

More can be said (in particular, it is not too hard to check that $w(n,2) = \dfrac{(n-2)(n+1)}{2}$ for any $n \geq 2$). Alas, there is no explicit formula for $w(n,k)$ in general. However, there is a recursion:

Proposition 4.4.5. For any positive integer n and any $k \in \mathbb{Z}$, we have

$$w(n,k) = \sum_{i=k-n+1}^{k} w(n-1,i).$$

Proof idea. We shall be sloppy and identify any permutation $\sigma \in S_n$ with its OLN $(\sigma(1), \sigma(2), \ldots, \sigma(n))$; thus, the "$i$-th entry" of σ shall mean $\sigma(i)$, and we shall be speaking of "removing an entry from a permutation" (meaning, of course, removing it from its OLN).

Fix a positive integer n and any $k \in \mathbb{Z}$. Then, if σ is a permutation of $[n]$, then removing the entry n from σ (that is, from the OLN of σ) yields a permutation of $[n-1]$ (because the remaining $n-1$ entries will be $1, 2, \ldots, n-1$ in some order). Let us denote the latter permutation of $[n-1]$ by $\overline{\sigma}$. For example, if $n = 5$ and $\sigma = 31254$ (in OLN), then $\overline{\sigma} = 3124 \in S_4$ (again in OLN).

Now, fix any $p \in [n]$. If $\sigma \in S_n$ is any permutation satisfying $\sigma(p) = n$, then:

[2] See [Grinbe15, Exercise 5.2 **(d)**] for the details of this step.

- We have $\ell(\overline{\sigma}) = \ell(\sigma) - (n-p)$. (Indeed, $\sigma(p) = n$ shows that the entry n lies in the p-th position of the OLN of σ, and therefore all the $n-p$ pairs
$$(p, p+1), \quad (p, p+2), \quad (p, p+3), \quad \ldots, \quad (p, n)$$
are inversions of σ. All these $n-p$ inversions are lost when we remove the entry n from σ. Any other inversions of σ are preserved when we remove the entry n from σ, although they can change slightly (specifically, if (i, j) is an inversion of σ with $i \neq p$, then it becomes (i', j') after removing the entry n from σ, where $i' := \begin{cases} i, & \text{if } i < p; \\ i-1, & \text{if } i > p \end{cases}$ and $j' := \begin{cases} j, & \text{if } j < p; \\ j-1, & \text{if } j > p \end{cases}$). Altogether, the permutation σ thus loses $n-p$ inversions when we remove the entry n from it. In other words, $\ell(\overline{\sigma}) = \ell(\sigma) - (n-p)$, since $\overline{\sigma}$ is precisely the result of removing the entry n from σ.)
- Hence, $\ell(\sigma) = k$ holds if and only if $\ell(\overline{\sigma}) = k - (n-p)$.
- We can uniquely reconstruct σ from $\overline{\sigma}$. (Indeed, $\overline{\sigma}$ is obtained from σ by removing the entry n, which lies in the p-th position of the OLN. Thus, in order to undo this removal, we simply need to reinsert n into the p-th position, shifting all later entries one position to the right.)

Thus, we obtain a map
$$\{\sigma \in S_n \mid \sigma(p) = n \text{ and } \ell(\sigma) = k\} \to \{\tau \in S_{n-1} \mid \ell(\tau) = k - (n-p)\},$$
$$\sigma \mapsto \overline{\sigma}.$$

This map is bijective[3]. Hence, the bijection principle yields
$$\begin{aligned} &(\text{\# of } \sigma \in S_n \text{ satisfying } \sigma(p) = n \text{ and } \ell(\sigma) = k) \\ &= (\text{\# of } \tau \in S_{n-1} \text{ satisfying } \ell(\tau) = k - (n-p)) \\ &= w(n-1,\ k-(n-p)) \end{aligned} \tag{4}$$
(by the definition of $w(n-1,\ k-(n-p))$).

Now, forget that we fixed p. We have thus proved (4) for each $p \in [n]$. However, for each permutation $\sigma \in S_n$, there is a unique $p \in [n]$ satisfying $\sigma(p) = n$. Thus, by the sum rule, we have
$$\begin{aligned} &(\text{\# of } \sigma \in S_n \text{ satisfying } \ell(\sigma) = k) \\ &= \sum_{p \in [n]} \underbrace{(\text{\# of } \sigma \in S_n \text{ satisfying } \sigma(p) = n \text{ and } \ell(\sigma) = k)}_{\substack{= w(n-1,\ k-(n-p)) \\ \text{(by (4))}}} \\ &= \sum_{p \in [n]} w(n-1,\ k-(n-p)) = \sum_{i=k-n+1}^{k} w(n-1, i) \end{aligned}$$

[3]Indeed, it is injective (since we can uniquely reconstruct σ from $\overline{\sigma}$, as we have already seen above) and surjective (since the just-mentioned method for reconstructing σ can be applied to any permutation $\tau \in S_{n-1}$ satisfying $\ell(\tau) = k - (n-p)$, and thus provides a permutation $\sigma \in S_n$ satisfying $\sigma(p) = n$ and $\ell(\sigma) = k$ and $\overline{\sigma} = \tau$).

(here, we have substituted i for $k-(n-p)$ in the sum). Since the left hand side of this equality is $w(n,k)$, we thus have proved Proposition 4.4.5. □

Equivalently, there is a **generating function formula** for the $w(n,k)$:

Proposition 4.4.6. Consider polynomials in an indeterminate X. For any $n \in \mathbb{N}$, we have

$$\sum_{\sigma \in S_n} X^{\ell(\sigma)}$$
$$= \prod_{i=1}^{n} \left(1 + X + X^2 + \cdots + X^{i-1}\right)$$
$$= (1+X)\left(1+X+X^2\right)\left(1+X+X^2+X^3\right)\cdots\left(1+X+X^2+\cdots+X^{n-1}\right).$$

Note that the LHS here is $\sum\limits_{k \in \mathbb{Z}} w(n,k) X^k$.

Proof idea. This can be proved by induction on n using Proposition 4.4.5. A different proof can be found in [21s, Proposition 5.3.5]. □

We remark that the distribution of the Coxeter lengths of all permutations $\sigma \in S_n$ is symmetric around $\dbinom{n}{2}/2$. In other words:

Remark 4.4.7. We have $w(n,k) = w\left(n, \dbinom{n}{2} - k\right)$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Proof idea. Nice (and easy) exercise. □

4.4.3. Products of simple transpositions

Theorem 4.3.23 **(b)** shows that any permutation σ of a finite set X can be written as a product of transpositions, and describes the smallest # of transpositions needed for this. In this subsection, we shall prove a similar result about **simple** transpositions (i.e., the transpositions $s_i = t_{i,i+1}$) instead of arbitrary transpositions. For this, of course, we need X to be $[n]$ (otherwise, simple transpositions aren't defined), and we should expect that writing σ as a product of simple transpositions will require more factors than writing σ as a product of transpositions (since simple transpositions are more restrictive).

This is indeed the case. In order to get to our result, we first need some properties of Coxeter lengths:

Proposition 4.4.8. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$. Then, $\ell\left(\sigma^{-1}\right) = \ell(\sigma)$.

Proof idea. Find the bijection! Or look it up in [21s, Proposition 5.3.13]. □

The next result is a crucial lemma, which characterizes how the Coxeter length of a permutation σ changes when we multiply σ by a simple transposition s_k (either from the left or from the right):

Lemma 4.4.9 (single swap lemma)**.** Let $n \in \mathbb{N}$. Let $\sigma \in S_n$ and $k \in [n-1]$. Then:

(a) We have

$$\ell(\sigma s_k) = \begin{cases} \ell(\sigma)+1, & \text{if } \sigma(k) < \sigma(k+1); \\ \ell(\sigma)-1, & \text{if } \sigma(k) > \sigma(k+1). \end{cases}$$

(b) We have

$$\ell(s_k \sigma) = \begin{cases} \ell(\sigma)+1, & \text{if } \sigma^{-1}(k) < \sigma^{-1}(k+1); \\ \ell(\sigma)-1, & \text{if } \sigma^{-1}(k) > \sigma^{-1}(k+1). \end{cases}$$

[**Note:** The OLN of σs_k is obtained from the OLN of σ by swapping the k-th and $(k+1)$-st entries, whereas the OLN of $s_k \sigma$ is obtained from the OLN of σ by swapping the entries k and $k+1$ (wherever they lie). The condition $\sigma(k) < \sigma(k+1)$ means that the k-th entry in the OLN of σ is smaller than the $(k+1)$-th entry. The condition $\sigma^{-1}(k) < \sigma^{-1}(k+1)$ means that the entry k is to the left of the entry $k+1$ in the OLN of σ.]

Proof idea. **(a)** A pair (i,j) will be called **active** if it is $(k,k+1)$, and **inactive** otherwise. Thus, there is exactly one active pair, which is $(k,k+1)$.

If $\sigma(k) < \sigma(k+1)$, then the active pair $(k,k+1)$ is an inversion of σs_k but not of σ. If $\sigma(k) > \sigma(k+1)$, then it is the other way round. Hence, we have

$$(\text{\# of active inversions of } \sigma) = \begin{cases} 0, & \text{if } \sigma(k) < \sigma(k+1); \\ 1, & \text{if } \sigma(k) > \sigma(k+1) \end{cases} \tag{5}$$

and

$$(\text{\# of active inversions of } \sigma s_k) = \begin{cases} 1, & \text{if } \sigma(k) < \sigma(k+1); \\ 0, & \text{if } \sigma(k) > \sigma(k+1). \end{cases} \tag{6}$$

What about the inactive pairs? We claim that the permutations σ and σs_k have the same # of inactive inversions. In fact, the permutation σs_k is obtained from σ by swapping the k-th and $(k+1)$-st entries in the OLN; thus, the k-th entry moves one position to its right, while the $(k+1)$-st entry moves one position to its left, and all remaining entries stay where they are. Hence, if a bigger entry comes before a smaller entry in the OLN of σ, then the former bigger entry will still come before the latter smaller entry in the OLN of σs_k, unless these two entries were the k-th and the $(k+1)$-st entries to begin with. In other words, any inactive inversion (i,j) of σ gives rise to an inactive inversion $(s_k(i), s_k(j))$ of σs_k (the "s_k"s in the latter inversion are coming from the fact that the k-th and the $(k+1)$-st entries got moved from σ to σs_k). Conversely, any inactive inversion of σs_k can be obtained in this way (why?). Thus, we obtain a bijection

$$\begin{aligned}\{\text{inactive inversions of } \sigma\} &\to \{\text{inactive inversions of } \sigma s_k\}, \\ (i,j) &\mapsto (s_k(i), s_k(j)).\end{aligned}$$

The bijection principle therefore yields

$$(\text{\# of inactive inversions of } \sigma) = (\text{\# of inactive inversions of } \sigma s_k). \tag{7}$$

Now, the definition of Coxeter length yields

$$\begin{aligned}\ell(\sigma s_k) &= (\text{\# of inversions of } \sigma s_k) \\ &= \underbrace{(\text{\# of active inversions of } \sigma s_k)}_{\substack{=\begin{cases}1, & \text{if } \sigma(k) < \sigma(k+1); \\ 0, & \text{if } \sigma(k) > \sigma(k+1)\end{cases} \\ \text{(by (6))}}} + \underbrace{(\text{\# of inactive inversions of } \sigma s_k)}_{\substack{=(\text{\# of inactive inversions of } \sigma) \\ \text{(by (7))}}} \\ &= \begin{cases}1, & \text{if } \sigma(k) < \sigma(k+1); \\ 0, & \text{if } \sigma(k) > \sigma(k+1)\end{cases} + (\text{\# of inactive inversions of } \sigma)\end{aligned}$$

and

$$\begin{aligned}\ell(\sigma) &= (\text{\# of inversions of } \sigma) \\ &= \underbrace{(\text{\# of active inversions of } \sigma)}_{\substack{=\begin{cases}0, & \text{if } \sigma(k) < \sigma(k+1); \\ 1, & \text{if } \sigma(k) > \sigma(k+1)\end{cases} \\ \text{(by (5))}}} + (\text{\# of inactive inversions of } \sigma) \\ &= \begin{cases}0, & \text{if } \sigma(k) < \sigma(k+1); \\ 1, & \text{if } \sigma(k) > \sigma(k+1)\end{cases} + (\text{\# of inactive inversions of } \sigma).\end{aligned}$$

The right hand sides of these two equalities differ by 1 or -1 depending on whether $\sigma(k) < \sigma(k+1)$ or $\sigma(k) > \sigma(k+1)$. Thus, so do the left hand sides. In other words,

$$\begin{aligned}\ell(\sigma s_k) &= \ell(\sigma) + \begin{cases}1, & \text{if } \sigma(k) < \sigma(k+1); \\ -1, & \text{if } \sigma(k) > \sigma(k+1)\end{cases} \\ &= \begin{cases}\ell(\sigma)+1, & \text{if } \sigma(k) < \sigma(k+1); \\ \ell(\sigma)-1, & \text{if } \sigma(k) > \sigma(k+1).\end{cases}\end{aligned}$$

This proves Lemma 4.4.9 **(a)**.

(b) Apply Lemma 4.4.9 **(a)** to σ^{-1} instead of σ, and translate back using Proposition 4.4.8 (and $s_k^{-1} = s_k$). □

The following theorem is a counterpart to Theorem 4.3.23 for **simple** transpositions (instead of arbitrary transpositions):

Theorem 4.4.10. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Then:

(a) We can write σ as a product of $\ell(\sigma)$ simple transpositions (which are not necessarily distinct!).

(b) The number $\ell(\sigma)$ is the smallest $p \in \mathbb{N}$ such that σ can be written as a product of p simple transpositions.

Example 4.4.11. Let $n = 5$, and let $\sigma \in S_5$ be the permutation whose OLN is 32415. Then, there are several ways to write σ as a product of 4 simple transpositions:

$$\sigma = s_1 s_3 s_2 s_1 = s_3 s_1 s_2 s_1 = s_3 s_2 s_1 s_2,$$

as well as infinitely many ways to write σ as a product of more than 4 simple transpositions (for example, $\sigma = s_2 s_3 s_2 s_3 s_1 s_2$). This is well in line with what Theorem 4.4.10 claims (since $\ell(\sigma) = 4$).

Proof idea for Theorem 4.4.10. What follows is but a brief sketch; see [21s, Theorem 5.3.17] for more details.

(a) Induct on $\ell(\sigma)$.

Base case: The case $\ell(\sigma) = 0$ follows from Proposition 4.4.3 **(b)**.

Induction step: If $\ell(\sigma) = h + 1 > 0$, then there exists some $k \in [n-1]$ such that $\sigma(k) > \sigma(k+1)$ (why?), and then Lemma 4.4.9 **(a)** shows that $\ell(\sigma s_k) = h$, which allows us to apply the induction hypothesis to σs_k instead of σ.

(b) Assume that $\sigma = s_{i_1} s_{i_2} \cdots s_{i_p}$ for some $i_1, i_2, \ldots, i_p$. We must show that $p \geq \ell(\sigma)$.

Set $\sigma_j := s_{i_1} s_{i_2} \cdots s_{i_j}$ for each $j \in \{0, 1, \ldots, p\}$. Argue (using Lemma 4.4.9 **(a)**) that $\ell(\sigma_j) \leq \ell(\sigma_{j-1}) + 1$ for each $j \in [p]$. Conclude that $\ell(\sigma_p) \leq \ell(\sigma_0) + p$. In view of $\sigma_p = \sigma$ and $\sigma_0 = \text{id}$, this can be simplified to $\ell(\sigma) \leq p$, qed. □

Note that Theorem 4.4.10 is the reason why the Coxeter length $\ell(\sigma)$ is called a "length": It really is the length of σ if we regard σ as a string of simple transpositions... And people often do view permutations in this way; in fact, this is the idea behind the Coxeter group approach to the symmetric group (see, e.g., [Willia03, Chapter 1] for an introduction).

Using Lemma 4.4.9 and Theorem 4.4.10 **(a)**, we can easily prove the following fact:

Proposition 4.4.12. Let $n \in \mathbb{N}$. Let $\alpha, \beta \in S_n$ be two permutations. Then:

(a) We have $\ell(\alpha\beta) \leq \ell(\alpha) + \ell(\beta)$.

(b) We have $\ell(\alpha\beta) \equiv \ell(\alpha) + \ell(\beta) \mod 2$. (In other words, the integers $\ell(\alpha\beta)$ and $\ell(\alpha) + \ell(\beta)$ are either both even or both odd.)

Proof idea. See [21s, Corollary 5.3.20 **(b)**] for part **(a)**, and [21s, Corollary 5.3.20 **(a)**] for part **(b)**.

(Here is the proof in a nutshell: Use Theorem 4.4.10 **(a)** to write β as a product of $\ell(\beta)$ simple transpositions: $\beta = s_{k_1} s_{k_2} \cdots s_{k_{\ell(\beta)}}$. Thus, the permutation $\alpha\beta$ can be obtained by successively multiplying α with the $\ell(\beta)$ simple transpositions $s_{k_1}, s_{k_2}, \ldots, s_{k_{\ell(\beta)}}$. However, when we multiply a permutation σ by a single simple transposition s_k (on the left or on the right), its Coxeter length $\ell(\sigma)$ either increases by 1 or decreases by 1 (according to Lemma 4.4.9). Hence, this length $\ell(\sigma)$ changes its parity (i.e., becomes even if it was odd, and becomes odd if it was even), and increases by at most 1. Therefore, when we successively multiply α with the $\ell(\beta)$ simple transpositions $s_{k_1}, s_{k_2}, \ldots, s_{k_{\ell(\beta)}}$, the length $\ell(\alpha)$ changes its parity $\ell(\beta)$ many times in total, and increases by at most $\ell(\beta)$ in total. This yields both parts of Proposition 4.4.12.) □

4.5. Signs of permutations

One of the most useful features of a permutation is its **sign** (aka **signature**). For a permutation of $[n]$, the easiest way to define the sign is as follows:

Definition 4.5.1. Let $n \in \mathbb{N}$. The **sign** of a permutation $\sigma \in S_n$ is defined to be the integer $(-1)^{\ell(\sigma)}$.

We denote it by $(-1)^{\sigma}$. (Other common notations for it are $\operatorname{sgn}(\sigma)$ or $\operatorname{sign}(\sigma)$ or $\varepsilon(\sigma)$. It is also known as the **signature** of σ.)

Thus, the sign $(-1)^{\sigma}$ of a permutation $\sigma \in S_n$ is 1 if its Coxeter length $\ell(\sigma)$ is even, and -1 if $\ell(\sigma)$ is odd. Thus, many properties of signs follow easily from properties of Coxeter lengths. Here is a collection of basic properties of signs:

Proposition 4.5.2. Let $n \in \mathbb{N}$.

(a) The sign of the permutation $\mathrm{id} \in S_n$ is $(-1)^{\mathrm{id}} = 1$.

(b) For any two distinct elements i and j of $[n]$, the transposition $t_{i,j} \in S_n$ has sign $(-1)^{t_{i,j}} = -1$.

(c) For any positive integer k and any distinct elements $i_1, i_2, \ldots, i_k \in [n]$, the k-cycle $\operatorname{cyc}_{i_1,i_2,\ldots,i_k}$ has sign $(-1)^{\operatorname{cyc}_{i_1,i_2,\ldots,i_k}} = (-1)^{k-1}$.

(d) We have $(-1)^{\alpha\beta} = (-1)^{\alpha} \cdot (-1)^{\beta}$ for any $\alpha \in S_n$ and $\beta \in S_n$.

(e) We have $(-1)^{\sigma_1\sigma_2\cdots\sigma_p} = (-1)^{\sigma_1}(-1)^{\sigma_2}\cdots(-1)^{\sigma_p}$ for any $\sigma_1, \sigma_2, \ldots, \sigma_p \in S_n$.

(f) We have $(-1)^{\sigma^{-1}} = (-1)^{\sigma}$ for any $\sigma \in S_n$. (The left hand side here has to be understood as $(-1)^{\left(\sigma^{-1}\right)}$.)

(g) We have

$$(-1)^{\sigma} = \prod_{1 \leq i < j \leq n} \frac{\sigma(i) - \sigma(j)}{i - j} \qquad \text{for each } \sigma \in S_n.$$

(The product sign "$\prod_{1 \leq i < j \leq n}$" means a product over all pairs (i, j) of integers satisfying $1 \leq i < j \leq n$. There are $\binom{n}{2}$ such pairs.)

(h) If $x_1, x_2, \ldots, x_n$ are any numbers, and if $\sigma \in S_n$, then

$$\prod_{1 \leq i < j \leq n} \left(x_{\sigma(i)} - x_{\sigma(j)} \right) = (-1)^{\sigma} \prod_{1 \leq i < j \leq n} \left(x_i - x_j \right).$$

(i) If $\sigma \in S_n$ is a permutation with exactly k orbits, then $(-1)^{\sigma} = (-1)^{n-k}$.

Proof sketch. **(a)** This is [Grinbe15, Proposition 5.15 **(a)**], and follows easily from $\ell(\mathrm{id}) = 0$.

(d) This is [Grinbe15, Proposition 5.15 **(c)**], and follows easily from Proposition 4.4.12 **(b)**. A different proof appears in [Strick13, Proposition B.13].

(e) This is [Grinbe15, Proposition 5.28], and follows by induction from Proposition 4.5.2 **(d)**.

(f) This is [Grinbe15, Proposition 5.15 **(d)**], and follows easily from Proposition 4.5.2 **(d)** or from Proposition 4.4.8.

(h) This is [Grinbe15, Exercise 5.13 **(a)**]. The proof is fairly easy: Each factor $x_{\sigma(i)} - x_{\sigma(j)}$ on the left hand side appears also on the right hand side, albeit with a different sign if (i, j) is an inversion of σ. Thus, the products on both sides agree up to a sign, which is precisely $(-1)^{\ell(\sigma)} = (-1)^{\sigma}$.

(g) This is [Grinbe15, Exercise 5.13 **(c)**], and is a particular case of Proposition 4.5.2 **(h)**.

(b) This is [Grinbe15, Exercise 5.10 **(b)**], but can also be proved rather directly: Let i and j be two distinct elements of $[n]$. Assume WLOG that $i < j$ (else, swap i with j). Then, the inversions of the transposition $t_{i,j}$ are the pairs (i, k) for all $k \in \{i+1, i+2, \ldots, j-1\}$, as well as the pairs (k, j) for all $k \in \{i+1, i+2, \ldots, j-1\}$, as well as the pair (i, j). The total # of these inversions is therefore $2(j-i-1)+1$. In other words, $\ell\left(t_{i,j}\right) = 2(j-i-1)+1$. Hence, $(-1)^{t_{i,j}} = (-1)^{\ell\left(t_{i,j}\right)} = (-1)^{2(j-i-1)+1} = -1$.

(c) This is [Grinbe15, Exercise 5.17 **(d)**], and follows easily from Proposition 4.5.2 **(e)** and from the fact that

$$\mathrm{cyc}_{i_1, i_2, \ldots, i_k} = t_{i_1, i_2} \circ t_{i_2, i_3} \circ \cdots \circ t_{i_{k-1}, i_k}$$

(check this!).

(i) This is [21s, Proposition 5.5.7] (since σ has k orbits if and only if σ has k cycles in its DCD). Alternatively, this follows easily from Proposition 4.5.2 **(e)** combined with Theorem 4.3.23 **(b)** (just keep in mind that our k here is not the k from Theorem 4.3.23 **(b)**). □

Using the signs, the permutations of $[n]$ can be classified into "even" and "odd" ones:

Definition 4.5.3. Let $n \in \mathbb{N}$. A permutation $\sigma \in S_n$ is said to be

- **even** if $(-1)^{\sigma} = 1$ (that is, if $\ell(\sigma)$ is even);
- **odd** if $(-1)^{\sigma} = -1$ (that is, if $\ell(\sigma)$ is odd).

We note that the even and the odd permutations of $[n]$ are equinumerous when $n \geq 2$:

Corollary 4.5.4. Let $n \geq 2$. Then,

$$(\text{\# of even permutations } \sigma \in S_n) = (\text{\# of odd permutations } \sigma \in S_n) = n!/2.$$

Proof idea. (See [Grinbe15, Exercise 5.4] for details.) Argue that the map

$$\{\text{even permutations } \sigma \in S_n\} \to \{\text{odd permutations } \sigma \in S_n\}, \\ \sigma \mapsto \sigma s_1$$

(this is the map that swaps the first two entries in the OLN of a permutation) is a bijection. □

The sign and the "parity" (i.e., evenness/oddness) of a permutation have applications all around mathematics. Most prominently, the **determinant** $\det A$ of an $n \times n$-matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

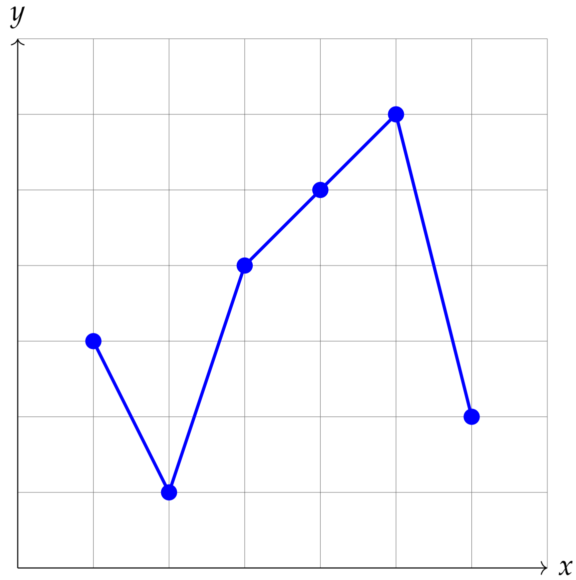
is defined as follows:

$$\det A := \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Note how the sign $(-1)^{\sigma}$ appears in this sum, making some of its addends positive and some negative. The sign of a permutation also appears as a useful invariant in the analysis of **permutation puzzles** (such as Rubik's cube and the 15-game; see [Mulhol21, Chapters 7–8 and Theorem 20.2.1] for example).

4.6. Descents and Eulerian numbers

Imagine you are drawing the "naive" plot of a permutation $\sigma \in S_n$ by marking the points $(1, \sigma(1))$, $(2, \sigma(2))$, ..., $(n, \sigma(n))$ on the Cartesian plane and connecting them with line segments. For instance, if σ is the permutation S_6 whose OLN is 314562, then this plot looks as follows:



(8)

(where both axes start at 0 and end at 7). This kind of plot can hardly be called combinatorially meaningful (after all, σ is just a map on a finite set, and it makes no combinatorial sense to "connect the dots" with line segments), but it serves a good purpose in making some properties of σ clear. In particular, we see that our map σ decreases on the interval $[1,2]$, then increases on $[2,5]$, then decreases again on $[5,6]$. This kind of increasing-decreasing behavior is typically subsumed under the notion of "descents":

Definition 4.6.1. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$ be a permutation. Then, a **descent** of σ means an $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$.

For example, the permutation $\sigma \in S_6$ that we have "plotted" in (8) has descents 1 (since $\sigma(1) > \sigma(2)$) and 5 (since $\sigma(5) > \sigma(6)$).

We can restate the definition of a descent as follows: A **descent** of a permutation $\sigma \in S_n$ is an $i \in [n-1]$ such that $(i, i+1)$ is an inversion of σ. Thus, descents can be viewed as special kinds of inversions (up to the fact that a descent is not a pair).

Just as we defined descents, we can define "ascents": An **ascent** of a permutation $\sigma \in S_n$ means an $i \in [n-1]$ such that $\sigma(i) < \sigma(i+1)$. Of course, the ascents of σ are just the elements of $[n-1]$ that are not descents of σ (since the injectivity of σ rules out the "third option" $\sigma(i) = \sigma(i+1)$). Thus, studying descents is equivalent to studying ascents.

Definition 4.6.2. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$ be a permutation. Then, the **descent set** of σ is defined to be the set of all descents of σ. It is denoted by $\operatorname{Des} \sigma$.

Thus,

$$\operatorname{Des} \sigma = \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$$

for each $\sigma \in S_n$. For instance, the permutation $\sigma \in S_6$ that we have "plotted" in (8) has descent set $\operatorname{Des} \sigma = \{1, 5\}$.

The concepts we have just introduced suggest some counting problems. We begin with some of the easy ones:

Exercise 1. Fix an integer $n \geq 4$.

(a) How many permutations $\sigma \in S_n$ satisfy $1 \in \operatorname{Des} \sigma$ (that is, $\sigma(1) > \sigma(2)$) ?

(b) How many permutations $\sigma \in S_n$ satisfy $1, 2 \in \operatorname{Des} \sigma$ (that is, $\sigma(1) > \sigma(2) > \sigma(3)$) ?

(c) How many permutations $\sigma \in S_n$ satisfy $1, 3 \in \operatorname{Des} \sigma$ (that is, $\sigma(1) > \sigma(2)$ and $\sigma(3) > \sigma(4)$) ?

Hints. Here are the rough ideas (see [18s-mt1s, §0.2] for the details).

(a) The answer is $\dfrac{n!}{2}$. Indeed, the # of permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2)$ equals the # of permutations $\sigma \in S_n$ satisfying $\sigma(1) < \sigma(2)$, since there is a bijection from the former permutations to the latter (given by swapping the first two values of σ, or, equivalently, by sending σ to $\sigma \circ t_{1,2}$). But the two #s clearly add up to $n!$, so they both must be $\dfrac{n!}{2}$.

(b) The answer is $\dfrac{n!}{6}$. Indeed, there are 6 ways in which the first three values $\sigma(1), \sigma(2), \sigma(3)$ of a permutation $\sigma \in S_n$ can be relatively ordered (namely, $\sigma(1) < \sigma(2) < \sigma(3)$ and $\sigma(1) < \sigma(3) < \sigma(2)$ and $\sigma(2) < \sigma(1) < \sigma(3)$ and $\sigma(2) < \sigma(3) < \sigma(1)$ and $\sigma(3) < \sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(2) < \sigma(1)$), and they are all equally likely (since there are bijections going between the corresponding permutations σ), so each of them appears exactly $\dfrac{n!}{6}$ times.

(c) The answer is $\dfrac{n!}{4}$. Indeed, there are 4 ways in which the two inequality signs in "$\sigma(1) \lesseqgtr \sigma(2)$" and "$\sigma(3) \lesseqgtr \sigma(4)$" may point, and they are all equally likely again. □

Here is a subtler counting problem: How many permutations $\sigma \in S_n$ have a given # of descents? The answer to this problem has a name:

Definition 4.6.3. Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then, the **Eulerian number** $\left\langle {n \atop k} \right\rangle$ is defined to be the # of all permutations $\sigma \in S_n$ that have exactly k descents (i.e., that satisfy $|\operatorname{Des} \sigma| = k$).

Example 4.6.4. We have $\left\langle {4 \atop 2} \right\rangle = 11$, since there are exactly 11 permutations $\sigma \in S_4$ that have exactly 2 descents. Indeed, these 11 permutations are (written in OLN)

1432,	2143,	2431,	3142,	3214,	3241,
3421,	4132,	4231,	4213,	4312.	

Here is a table of some Eulerian numbers:

$\left\langle {n \atop k} \right\rangle$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$k=0$	1	1	1	1	1	1	1	1
$k=1$	0	0	1	4	11	26	57	120
$k=2$	0	0	0	1	11	66	302	1191
$k=3$	0	0	0	0	1	26	302	2416
$k=4$	0	0	0	0	0	1	57	1191
$k=5$	0	0	0	0	0	0	1	120
$k=6$	0	0	0	0	0	0	0	1
$k=7$	0	0	0	0	0	0	0	0

(see the Wikipedia page for more). Here are some basic properties:

Proposition 4.6.5. **(a)** We have $\left\langle {n \atop k} \right\rangle = 0$ for any $n, k \in \mathbb{N}$ satisfying $k \geq n$ and $k > 0$.

(b) We have $\left\langle {n \atop 0} \right\rangle = 1$ for any $n \in \mathbb{N}$.

(c) We have $\left\langle {n \atop n-1} \right\rangle = 1$ for any $n > 0$.

(d) We have $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$ for any integer $n > 0$ and any $k \in \mathbb{Z}$.

(e) We have $\left\langle {n \atop 1} \right\rangle = 2^n - (n+1)$ for any $n > 0$.

(f) We have $\left\langle {n \atop k} \right\rangle = 0$ for any $n \in \mathbb{N}$ and any negative $k \in \mathbb{Z}$.

Proof idea. **(a)** A permutation $\sigma \in S_n$ cannot have more than $n-1$ descents unless $n = 0$ (since $\operatorname{Des} \sigma \subseteq [n-1]$). Thus, part **(a)** follows.

(f) This is even more obvious than part **(a)**.

(b) A permutation $\sigma \in S_n$ that has 0 descents must satisfy $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(n)$ and therefore $\sigma = \operatorname{id}$ (why?[4]). Thus, there is exactly 1 such permutation σ. Hence, $\left\langle {n \atop 0} \right\rangle = 1$.

(d) (See [18f-hw4s, Exercise 3] for details.) Let $n > 0$ and $k \in \mathbb{Z}$. Let w_0 be the permutation in S_n that sends each $i \in [n]$ to $n+1-i$. In other words, w_0 is the permutation in S_n whose OLN is $(n, n-1, n-2, \ldots, 2, 1)$. Then, it is easy to see that the descents of a permutation $\sigma \in S_n$ are precisely the ascents of the permutation $w_0 \circ \sigma$ (since $\sigma(i) > \sigma(i+1)$ is equivalent to $(w_0 \circ \sigma)(i) < (w_0 \circ \sigma)(i+1)$). Hence, a

[4]See [Grinbe15, Exercise 5.2 **(d)**] for a detailed proof.

permutation $\sigma \in S_n$ has k descents if and only if the permutation $w_0 \circ \sigma$ has k ascents, i.e., has $n-1-k$ descents. Thus, we obtain a map

$$\text{from } \{\text{permutations } \sigma \in S_n \text{ that have } k \text{ descents}\}$$
$$\text{to } \{\text{permutations } \sigma \in S_n \text{ that have } n-1-k \text{ descents}\},$$

which sends each σ to $w_0 \circ \sigma$. This map is a bijection (why?). The bijection principle thus yields

$$\begin{aligned} &(\text{\# of all permutations } \sigma \in S_n \text{ that have } k \text{ descents}) \\ &= (\text{\# of all permutations } \sigma \in S_n \text{ that have } n-1-k \text{ descents}). \end{aligned}$$

In other words, $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$. This proves Proposition 4.6.5 **(d)**.

(c) Let $n > 0$. Then, Proposition 4.6.5 **(d)** yields $\left\langle {n \atop n-1} \right\rangle = \left\langle {n \atop n-1-(n-1)} \right\rangle = \left\langle {n \atop 0} \right\rangle = 1$ (by Proposition 4.6.5 **(b)**).

(e) Let $n > 0$. Then, the definition of $\left\langle {n \atop 1} \right\rangle$ yields

$$\begin{aligned} \left\langle {n \atop 1} \right\rangle &= (\text{\# of all permutations } \sigma \in S_n \text{ having exactly 1 descent}) \\ &= \sum_{i=1}^{n-1} (\text{\# of all permutations } \sigma \in S_n \text{ with } \operatorname{Des} \sigma = \{i\}) \end{aligned}$$

(by the sum rule). Now, let us compute the addends on the right hand side.

Indeed, let $i \in [n-1]$. How does a permutation $\sigma \in S_n$ with $\operatorname{Des} \sigma = \{i\}$ look like? Such a permutation σ must satisfy the chain of inequalities

$$\sigma(1) < \sigma(2) < \cdots < \sigma(i) > \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$$

(all inequality signs in this chain are $<$ except for the sign between $\sigma(i)$ and $\sigma(i+1)$). This chain of inequalities is moreover necessary and sufficient for $\operatorname{Des} \sigma = \{i\}$.

Now, how many permutations $\sigma \in S_n$ satisfy this chain of inequalities? Let us first ignore the $>$ sign between $\sigma(i)$ and $\sigma(i+1)$. This breaks our chain into two chains $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$. The # of all permutations $\sigma \in S_n$ satisfying both $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$ is $\dbinom{n}{i}$ [5]. Only one of these $\dbinom{n}{i}$ permutations satisfies

[5] *Proof.* Any such permutation σ can be uniquely constructed by the following procedure:

1. Choose the i-element subset $\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ of $[n]$. (There are $\dbinom{n}{i}$ options for this subset.)
2. Let $\sigma(1), \sigma(2), \ldots, \sigma(i)$ be the i elements of this subset in increasing order.
3. Let $\sigma(i+1), \sigma(i+2), \ldots, \sigma(n)$ be the $n-i$ elements of $[n]$ that don't belong to this subset (again listed in increasing order).

$\sigma(i) \leq \sigma(i+1)$ (namely, the identity permutation id). Hence, the remaining $\binom{n}{i} - 1$ permutations satisfy $\sigma(i) > \sigma(i+1)$ and therefore

$$\sigma(1) < \sigma(2) < \cdots < \sigma(i) > \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n).$$

Hence,

$$\begin{aligned}&(\text{\# of all permutations } \sigma \in S_n \text{ satisfying the chain of}\\ &\qquad \text{inequalities } \sigma(1) < \sigma(2) < \cdots < \sigma(i) > \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n))\\ &= \binom{n}{i} - 1.\end{aligned}$$

In other words,

$$\begin{aligned}&(\text{\# of all permutations } \sigma \in S_n \text{ with } \operatorname{Des}\sigma = \{i\})\\ &= \binom{n}{i} - 1 \end{aligned} \tag{9}$$

(since we have previously observed that the chain of inequalities $\sigma(1) < \sigma(2) < \cdots < \sigma(i) > \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$ is necessary and sufficient for $\operatorname{Des}\sigma = \{i\}$).

Forget that we fixed i. We thus have proved (9) for each $i \in [n-1]$. Now, as we know,

$$\begin{aligned}\left\langle {n \atop 1} \right\rangle &= \sum_{i=1}^{n-1} \underbrace{(\text{\# of all permutations } \sigma \in S_n \text{ with } \operatorname{Des}\sigma = \{i\})}_{\substack{= \binom{n}{i} - 1 \\ \text{(by (9))}}}\\ &= \sum_{i=1}^{n-1} \left(\binom{n}{i} - 1 \right) = \underbrace{\sum_{i=1}^{n-1} \binom{n}{i}}_{= \sum_{i=0}^{n} \binom{n}{i} - \binom{n}{0} - \binom{n}{n}} - (n-1)\\ &= \underbrace{\sum_{i=0}^{n} \binom{n}{i}}_{\substack{=2^n \\ \text{(by Corollary 1.3.32} \\ \text{in Lecture 7)}}} - \underbrace{\binom{n}{0}}_{=1} - \underbrace{\binom{n}{n}}_{=1} - (n-1) = 2^n - 1 - 1 - (n-1)\\ &= 2^n - (n+1).\end{aligned}$$

This proves Proposition 4.6.5 **(e)**. □

The Eulerian numbers also satisfy a recursion:

Proposition 4.6.6. For any positive integer n and any integer k, we have

$$\left\langle {n \atop k} \right\rangle = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle.$$

Proof idea. See [18f-mt2s, Exercise 1 **(c)**]. (Ignore the assumption that k be positive; this assumption is not actually used.) □

More surprisingly, there is an explicit formula for the Eulerian numbers:

Theorem 4.6.7. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,

$$\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k+1-i)^n .$$

Proof sketch. (The below argument is taken from [Bona22, proof of Theorem 1.11]; it is due to Hugh Thomas and Richard Stanley.)

A k-**barper** (this is short for "k-barred permutation") shall mean an n-tuple containing each of the numbers $1, 2, \ldots, n$ exactly once (i.e., the one-line notation of some $\sigma \in S_n$), with (altogether) k **bars** placed between some of its entries (or at the very start, or at the very end). The bars subdivide the n-tuple into $k+1$ (possibly empty) **compartments**, each of which consists of the entries between two consecutive bars (or between the beginning of the n-tuple and the very first bar, or between the very last bar and the end of the n-tuple).

For example, for $n = 8$, here are some examples of 5-barpers:

$$\begin{aligned}
\beta_1 &= 5 \mid 1\,3 \mid 8 \mid \mid 2\,4\,6 \mid 7; \\
\beta_2 &= 5 \mid 1\,3\,8 \mid 2\,4 \ \mid \mid \mid \ 6\,7; \\
\beta_3 &= 5 \mid 1\,8\,3 \mid 2\,4 \ \mid \mid \ 6\,7 \ \mid; \\
\beta_4 &= \mid \mid 5 \mid 1\,3\,8 \mid 2\,4\,6\,7 \ \mid; \\
\beta_5 &= 5\,1 \mid 3\,8\,2 \ \mid \mid 4 \mid 6 \mid 7
\end{aligned}$$

(we are omitting the commas between the entries of the tuples, and the parentheses around the tuples). We note that several bars can be placed between two consecutive entries (or at the start, or at the end). The six compartments of β_2 are (5), $(1,3,8)$, $(2,4)$, $()$, $()$ and $(6,7)$.

A compartment of a k-barper is said to be **good** if its entries are in increasing order. In the above example, the second compartment of β_3 is not good (since its entries 1 8 3 are not in increasing order), and the first two compartments of β_5 are not good (since $5 > 1$ and $8 > 2$), but all other compartments are good.

A k-barper is said to be **good** if all its compartments are good. In the above example, the k-barpers $\beta_1, \beta_2, \beta_4$ are good, whereas β_3 and β_5 are not.

We observe that

$$(\text{\# of good } k\text{-barpers}) = (k+1)^n . \tag{10}$$

(Indeed, in order to construct a good k-barper, we only need to decide, for each $i \in [n]$, which of the $k+1$ compartments we want to place i in. The "goodness" then takes care of the ordering of the entries in each compartment.)

Some more notations are in order. If β is a good k-barper, then:

- A **wall** of β means a bar of β that is not immediately followed by another bar. In other words, a wall of β means a bar of β that marks the beginning of a nonempty compartment or has no walls to its right. For example, in the good k-barper β_2 from the above examples, the first, second and fifth bars are walls, whereas the third and fourth bars are not. In the good k-barper β_4, all bars except for the first are walls.

- A **useless wall** of β means a wall of β such that removing this wall yields a good $(k-1)$-barper. In other words, a useless wall of β means a wall of β such that each entry in the compartment just to its left is smaller than each entry in the compartment just to its right. For example, in the 5-barper $\beta_1 = 5 \mid 1\ 3 \mid 8 \mid \mid 2\ 4\ 6 \mid 7$ from the above examples, the second wall is useless (since removing it yields $5 \mid 1\ 3\ 8 \mid \mid 2\ 4\ 6 \mid 7$, which is a good 4-barper), but the first wall is not (since removing it yields $5\ 1\ 3 \mid 8 \mid \mid 2\ 4\ 6 \mid 7$, which is not good). Likewise, the third wall (= the fourth bar) in β_1 is useless (since removing it yields $5 \mid 1\ 3 \mid 8 \mid 2\ 4\ 6 \mid 7$, which is a good 4-barper), and so is the fourth wall (= the fifth bar). Note, in particular, that a wall placed at the very end or the very start of the tuple is automatically useless.

Now, we claim that

$$\left\langle {n \atop k} \right\rangle = (\text{\# of good } k\text{-barpers with no useless walls}) . \qquad (11)$$

[*Proof of (11):* If $\sigma \in S_n$ is a permutation having exactly k descents, then we can turn the OLN of this permutation σ into a k-barper β_σ by putting a bar "at each descent of σ" (that is, for each descent i of σ, we put a bar between $\sigma(i)$ and $\sigma(i+1)$ in the OLN of σ). For example, if $\sigma \in S_7$ is the permutation with OLN 4261375, then the resulting k-barper β_σ will be $4 \mid 2\ 6 \mid 1\ 3\ 7 \mid 5$. Note that the k-barper β_σ is good (since $\sigma(i) < \sigma(i+1)$ whenever i is not a descent of σ) and has no useless walls (since $\sigma(i) > \sigma(i+1)$ for each descent i of σ). Thus, we obtain a map

$$\begin{aligned} &\text{from } \{\text{permutations } \sigma \in S_n \text{ having exactly } k \text{ descents}\} \\ &\text{to } \{\text{good } k\text{-barpers with no useless walls}\} \end{aligned}$$

which sends each σ to β_σ. This map is easily seen to be a bijection, because a good k-barper with no useless walls can always be read as the OLN of a permutation $\sigma \in S_n$ having exactly k descents (just drop the bars)[6]. Thus, the bijection principle yields

$$\begin{aligned} &(\text{\# of permutations } \sigma \in S_n \text{ having exactly } k \text{ descents}) \\ &= (\text{\# of good } k\text{-barpers with no useless walls}) . \end{aligned}$$

But this is precisely (11), since the LHS equals $\left\langle {n \atop k} \right\rangle$.]

Now, we shall compute the RHS of (11) using the Principle of Inclusion and Exclusion.

[6]To see this, it helps to observe that if a k-barper β has no useless walls, then each bar of β is a wall (since otherwise, the next wall to the right of this bar would be useless), and there are no bars at the very beginning or the very end of β (for the same reason).

Let

$$U := \{\text{all good } k\text{-barpers}\}.$$

For each $i \in [n+1]$, let A_i be the set of all good k-barpers that have a useless wall between the $(i-1)$-st and the i-th entries[7]. Then,

$$\{\text{all good } k\text{-barpers with no useless walls}\} \\ = U \setminus (A_1 \cup A_2 \cup \cdots \cup A_{n+1}),$$

so that

$$(\text{\# of good } k\text{-barpers with no useless walls}) \\ = |U \setminus (A_1 \cup A_2 \cup \cdots \cup A_{n+1})| \\ = \sum_{I \subseteq [n+1]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|$$

(by Theorem 2.7.6 in Lecture 19). Hence, (11) becomes

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \sum_{I \subseteq [n+1]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|. \qquad (12)$$

Now, we shall compute the sizes on the RHS of this equality. We claim that

$$\left| \bigcap_{i \in I} A_i \right| = \begin{cases} (k - |I| + 1)^n, & \text{if } |I| \leq k; \\ 0, & \text{if } |I| > k \end{cases} \qquad (13)$$

for every subset I of $[n+1]$.

[*Proof of (13):* Let I be a subset of $[n+1]$. Then, $\bigcap_{i \in I} A_i$ is the set of all good k-barpers that have a useless wall between the $(i-1)$-st and the i-th entries for each $i \in I$. [8] Obviously, such k-barpers can exist only if $|I| \leq k$ (since a k-barper has only k bars, thus at most k useless walls). Thus, the set $\bigcap_{i \in I} A_i$ is empty when $|I| > k$. In other words, $\left| \bigcap_{i \in I} A_i \right| = 0$ when $|I| > k$. It remains to prove that $\left| \bigcap_{i \in I} A_i \right| = (k - |I| + 1)^n$ when $|I| \leq k$.

Let us thus assume that $|I| \leq k$. Thus, $k - |I| \in \mathbb{N}$.

Let $\beta \in \bigcap_{i \in I} A_i$ be arbitrary. Thus, β is a good k-barper that has a useless wall between the $(i-1)$-st and the i-th entries for each $i \in I$. If we delete all these $|I|$ many useless walls from β (leaving all other useless walls in place, if there are any), then we obtain

[7] If $i = 1$, then this means having a useless wall before the first entry. If $i = n+1$, then this means having a useless wall at the very end.

[8] Note that a good k-barper cannot have more than one wall between the same two adjacent entries (since only the last bar between these two entries counts as a wall).

a good $(k - |I|)$-barper[9], which we shall call β'. This yields a map

$$\text{from } \bigcap_{i \in I} A_i \text{ to } \{\text{good } (k - |I|) \text{-barpers}\}$$

that sends each $\beta \in \bigcap_{i \in I} A_i$ to the respective β'. This map is injective (since we can always reinsert the $|I|$ useless walls that we removed[10]) and surjective (since we can always start with any good $(k - |I|)$-barper and insert a wall between its $(i-1)$-st and the i-th entries for each $i \in I$; then, all these inserted walls will be useless[11], and the resulting k-barper will belong to $\bigcap_{i \in I} A_i$). Hence, it is bijective. The bijection principle therefore yields

$$\left| \bigcap_{i \in I} A_i \right| = (\#\text{ of good } (k - |I|) \text{-barpers}) = (k - |I| + 1)^n$$

(by (10), applied to $k - |I|$ instead of k). This completes the proof of (13).]

[9]This is not completely obvious. By definition, if we delete a single useless wall from β, then the resulting $(k-1)$-barper is good. But what we need to show is that if we delete j useless walls at the same time, then the resulting $(k-j)$-barper is still good. To convince yourself of this, observe that if the i-th entry of β is larger than the $(i+1)$-st entry, then these two entries are separated in β by a bar that is not a useless wall (i.e., either by a non-useless wall, or by a bar that is not a wall), and therefore they cannot end up in the same compartment upon removal of several useless walls.

[10]The positions at which they have to be reinserted are uniquely determined, since a wall separating two adjacent entries must always stand to the right of all bars separating these two entries.

[11]since the original $(k - |I|)$-barper was good

Now, (12) becomes[12]

$$\begin{aligned}
\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle &= \sum_{I \subseteq [n+1]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| \\
&= \sum_{I \subseteq [n+1]} (-1)^{|I|} \begin{cases} (k - |I| + 1)^n, & \text{if } |I| \leq k; \\ 0, & \text{if } |I| > k \end{cases} \qquad \text{(by (13))} \\
&= \sum_{i=0}^{n+1} \sum_{\substack{I \subseteq [n+1]; \\ |I| = i}} \underbrace{(-1)^{|I|} \begin{cases} (k - |I| + 1)^n, & \text{if } |I| \leq k; \\ 0, & \text{if } |I| > k \end{cases}}_{\substack{= (-1)^i \begin{cases} (k-i+1)^n, & \text{if } i \leq k; \\ 0, & \text{if } i > k \end{cases} \\ (\text{since } |I| = i)}} \\
&\qquad \left(\begin{array}{c} \text{here, we have split up the sum} \\ \text{according to the value of } |I| \end{array} \right) \\
&= \sum_{i=0}^{n+1} \underbrace{\sum_{\substack{I \subseteq [n+1]; \\ |I| = i}} (-1)^i \begin{cases} (k-i+1)^n, & \text{if } i \leq k; \\ 0, & \text{if } i > k \end{cases}}_{\substack{= \binom{n+1}{i} (-1)^i \begin{cases} (k-i+1)^n, & \text{if } i \leq k; \\ 0, & \text{if } i > k \end{cases} \\ (\text{since the \# of } i\text{-element subsets of } [n+1] \text{ is } \binom{n+1}{i})}} \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i \begin{cases} (k-i+1)^n, & \text{if } i \leq k; \\ 0, & \text{if } i > k \end{cases} \\
&\overset{0}{=} \sum_{i \in \mathbb{N}} \binom{n+1}{i} (-1)^i \begin{cases} (k-i+1)^n, & \text{if } i \leq k; \\ 0, & \text{if } i > k \end{cases} \qquad \left(\begin{array}{c} \text{since } \binom{n+1}{i} = 0 \\ \text{for all } i > n+1 \end{array} \right) \\
&= \sum_{i=0}^{k} \binom{n+1}{i} (-1)^i (k-i+1)^n + \underbrace{\sum_{i>k} \binom{n+1}{i} (-1)^i 0}_{=0} \\
&\qquad \left(\begin{array}{c} \text{here, we have split up the sum into one part} \\ \text{containing all addends with } i \leq k \text{, and another} \\ \text{part containing all addends with } i > k \end{array} \right) \\
&= \sum_{i=0}^{k} \binom{n+1}{i} (-1)^i (k-i+1)^n = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k+1-i)^n .
\end{aligned}$$

This proves Theorem 4.6.7. □

A different proof of Theorem 4.6.7 (using generating functions) appears in [Peters15, Corollary 1.3].

[12] I will be using the "$\overset{0}{=}$" symbol; its meaning has been explained in Remark 2.3.2 in Lecture 15.

See [Stanle11, §1.4 and §1.5], [Bona22, §1.1] and [Rzadko08] for more on descents and Eulerian numbers[13]. (In particular, [Stanle11, Proposition 1.4.1] generalizes Exercise 1.) See also [BayDia92] for an application of descents to probability theory (specifically, to the question of how a deck of cards looks like when it has been shuffled several times via a riffle shuffle).

There is much more to say about permutations, which we don't have time for in this short course:

- the permutahedron (e.g., [21s, Remark 5.3.19], [Santmy07]);
- Lehmer codes ([21s, §5.3.2]) and an **explicit** way to write a permutation $\sigma \in S_n$ as a product of $\ell(\sigma)$ simple transpositions ([21s, Remark 5.3.23 and Proposition 5.3.24]);
- pattern avoidance ([Bona22, Chapter 4]);
- Coxeter groups and reflection groups as generalizations of symmetric groups ([BjoBre05]);
- and much more...

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[13] Notational warning: The descent set $\operatorname{Des}\sigma$ is called $D(\sigma)$ in [Stanle11] and in [Bona22]. The Eulerian number $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ is called $A(n, k+1)$ in [Stanle11] and in [Bona22].

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The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2022-09-15c .

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