Math 222 Fall 2022, Lecture 25: The twelvefold way

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

3. The twelvefold way

3.5. $L \to U$ placements (cont'd)

We continue the study of $L \to U$ placements. Last time (Proposition 3.5.1 in Lecture 24), we proved that

$$(\text{\# of injective } L \to U \text{ placements } A \to X) = [|A| \leq |X|]$$

(where, as we recall, A is the set of balls and X is the set of boxes).

To count arbitrary and surjective $L \to U$ placements, we need a definition that we already mentioned in Lecture 17 (Remark 2.4.14):

Definition 3.5.2. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then, we define the **Stirling number of the second kind** $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ to be $\dfrac{\operatorname{sur}(n,k)}{k!}$.

Now we claim:

Proposition 3.5.3. We have

$$\begin{aligned} &(\text{\# of surjective } L \to U \text{ placements } A \to X) \\ &= \left\{ \begin{matrix} |A| \\ |X| \end{matrix} \right\} = \frac{\operatorname{sur}(|A|, |X|)}{|X|!}. \end{aligned}$$

Proof. First, we recall that $L \to U$ placements are $\overset{\text{box}}{\sim}$-equivalence classes of maps from A to X. Each $\overset{\text{box}}{\sim}$-equivalence class either consists entirely of surjections, or contains no surjection at all. (This is because two box-equivalent maps are either both surjective or both non-surjective.)

Every surjection[1] (and, more generally, every map from A to X) lies in exactly one $\overset{\text{box}}{\sim}$-equivalence class (since $\overset{\text{box}}{\sim}$ is an equivalence relation).

Now, the crux of our proof will be the following claim:

> *Claim 1:* Each $\overset{\text{box}}{\sim}$-equivalence class of surjections (i.e., each surjective $L \to U$ placement) contains exactly $|X|!$ many maps from A to X.

[1] Throughout this proof, "surjection" means "surjection from A to X".

[*Proof of Claim 1:* Consider the $\overset{\text{box}}{\sim}$-equivalence class of some surjection $g : A \to X$. The elements of this class are all the maps that are box-equivalent to g. In other words, the elements of this class are all the maps of the form $\xi \circ g$ with ξ being a permutation of X. There are $|X|!$ many permutations ξ of X, and they all lead to **distinct** maps $\xi \circ g$ [2]. Thus, there are exactly $|X|!$ many distinct maps $\xi \circ g$ in the class of g. This proves Claim 1.]

Now, Proposition 2.4.9 in Lecture 16 yields

$$(\text{\# of surjections from } A \text{ to } X) = \operatorname{sur}(|A|, |X|).$$

Hence,

$$\begin{aligned}
\operatorname{sur}(|A|, |X|) &= (\text{\# of surjections from } A \text{ to } X) \\
&= \sum_{\substack{C \text{ is a } \overset{\text{box}}{\sim}\text{-equivalence} \\ \text{class of surjections}}} \underbrace{(\text{\# of surjections in } C)}_{\substack{=|X|! \\ \text{(by Claim 1)}}} \\
&\qquad \left(\begin{array}{c}\text{by the sum rule, since each surjection} \\ \text{lies in exactly one } \overset{\text{box}}{\sim} \text{-equivalence class}\end{array}\right) \\
&= \sum_{\substack{C \text{ is a } \overset{\text{box}}{\sim}\text{-equivalence} \\ \text{class of surjections}}} |X|! \\
&= \left(\text{\# of } \overset{\text{box}}{\sim} \text{-equivalence classes of surjections}\right) \cdot |X|!.
\end{aligned}$$

Dividing this by $|X|!$, we obtain

$$\left(\text{\# of } \overset{\text{box}}{\sim} \text{-equivalence classes of surjections}\right) = \frac{\operatorname{sur}(|A|, |X|)}{|X|!} = \left\{ \begin{matrix} |A| \\ |X| \end{matrix} \right\}$$

(by the definition of $\left\{ \begin{matrix} |A| \\ |X| \end{matrix} \right\}$). Since the $\overset{\text{box}}{\sim}$-equivalence classes of surjections are precisely the surjective $L \to U$ placements, we thus have proved that

$$(\text{\# of surjective } L \to U \text{ placements}) = \left\{ \begin{matrix} |A| \\ |X| \end{matrix} \right\}.$$

This proves Proposition 3.5.3. $\square$

[2] *Proof.* Let ξ_1 and ξ_2 be two distinct permutations of X. We must show that the maps $\xi_1 \circ g$ and $\xi_2 \circ g$ are distinct.

Indeed, there exists some box $x \in X$ such that $\xi_1(x) \neq \xi_2(x)$ (since ξ_1 and ξ_2 are distinct). Consider such an x. Since g is surjective, there exists a ball $a \in A$ such that $g(a) = x$. Consider such an a. Now,

$$(\xi_1 \circ g)(a) = \xi_1 \left(\underbrace{g(a)}_{=x} \right) = \xi_1(x) \neq \xi_2 \left(\underbrace{x}_{=g(a)} \right) = \xi_2(g(a)) = (\xi_2 \circ g)(a).$$

This shows that $\xi_1 \circ g$ and $\xi_2 \circ g$ are distinct, qed.

The above proof is an instance of what is often called the "shepherd's principle": To count the sheep in a flock, count the legs and divide by 4. (In our case, the surjective $L \to U$ placements are the sheep; the actual surjections inside them are their legs; and each sheep has $|X|!$ many legs. This works well when each sheep has the same # of legs.)

It remains to count **all** $L \to U$ placements (as opposed to just the injective or surjective ones). This isn't very hard any more:

Proposition 3.5.4. We have

$$\begin{aligned}&(\text{\# of } L \to U \text{ placements from } A \text{ to } X)\\ &= \left\{ {|A| \atop 0} \right\} + \left\{ {|A| \atop 1} \right\} + \cdots + \left\{ {|A| \atop |X|} \right\} = \sum_{k=0}^{|X|} \left\{ {|A| \atop k} \right\}.\end{aligned}$$

Proof. The main step is to prove the following:

Claim 1: Let $k \in \{0, 1, \ldots, |X|\}$. Then,

$$\begin{aligned}&(\text{\# of } L \to U \text{ placements from } A \text{ to } X \text{ with exactly } k \text{ nonempty boxes})\\ &= \left\{ {|A| \atop k} \right\}.\end{aligned}$$

[*Proof of Claim 1:* The empty boxes in an $L \to U$ placement are indistinguishable (since the boxes are unlabelled) and can always be moved to the end of the placement. Hence, an $L \to U$ placement from A to X with exactly k nonempty boxes can be viewed as a surjective $L \to U$ placement from A to $[k]$ (since we can simply remove the empty boxes and rename the k nonempty boxes as $1, 2, \ldots, k$). Therefore,

$$\begin{aligned}&(\text{\# of } L \to U \text{ placements from } A \text{ to } X \text{ with exactly } k \text{ nonempty boxes})\\ &= (\text{\# of surjective } L \to U \text{ placements from } A \text{ to } [k])\\ &= \left\{ {|A| \atop k} \right\} \qquad (\text{by Proposition 3.5.3, applied to } [k] \text{ instead of } X).\end{aligned}$$

This proves Claim 1.]

Now, note that any $L \to U$ placement from A to X has exactly k nonempty boxes for a unique $k \in \{0, 1, \ldots, |X|\}$ (indeed, it cannot have more than $|X|$

nonempty boxes, since we have only $|X|$ boxes). Thus, by the sum rule, we find

$$\begin{aligned}
&(\text{\# of } L \to U \text{ placements from } A \text{ to } X) \\
&= \sum_{k=0}^{|X|} \underbrace{(\text{\# of } L \to U \text{ placements from } A \text{ to } X \text{ with exactly } k \text{ nonempty boxes})}_{\substack{= \left\{ \begin{matrix} |A| \\ k \end{matrix} \right\} \\ \text{(by Claim 1)}}} \\
&= \sum_{k=0}^{|X|} \left\{ \begin{matrix} |A| \\ k \end{matrix} \right\} = \left\{ \begin{matrix} |A| \\ 0 \end{matrix} \right\} + \left\{ \begin{matrix} |A| \\ 1 \end{matrix} \right\} + \cdots + \left\{ \begin{matrix} |A| \\ |X| \end{matrix} \right\}.
\end{aligned}$$

This proves Proposition 3.5.4. □

With the above three results proved, our twelvefold way table now looks as follows:

	arbitrary	injective	surjective
$L \to L$	$\|X\|^{\|A\|}$	$\|X\|^{\underline{\|A\|}}$	$\operatorname{sur}(\|X\|, \|A\|)$
$U \to L$	$\dbinom{\|A\| + \|X\| - 1}{\|A\|}$	$\dbinom{\|X\|}{\|A\|}$	$\dbinom{\|A\| - 1}{\|A\| - \|X\|}$
$L \to U$	$\sum_{k=0}^{\|X\|} \left\{ \begin{matrix} \|A\| \\ k \end{matrix} \right\}$	$[\|A\| \leq \|X\|]$	$\left\{ \begin{matrix} \|A\| \\ \|X\| \end{matrix} \right\}$
$U \to U$			

.

3.6. Set partitions

Proposition 3.5.3 gave us an example of a combinatorial object (surjective $L \to U$ placements) that is counted by Stirling numbers of the second kind. Let us mention another such object: the **set partitions**. In truth, these set partitions are in an easy bijection with surjective $L \to U$ placements, so they won't be of much novelty to us, but they have a tendency to appear in various places in mathematics, so it is worth introducing them anyway.

Definition 3.6.1. Let S be a set.

(a) A **set partition** of S is a set $\mathcal{F}$ of disjoint nonempty subsets of S such that the union of these subsets is S.

In other words, a set partition of S is a set $\{S_1, S_2, \ldots, S_k\}$ of nonempty subsets of S such that each element of S lies in exactly one of $S_1, S_2, \ldots, S_k$. (Here, we are assuming that S is finite; otherwise, we would have to allow "infinite" values of k.)

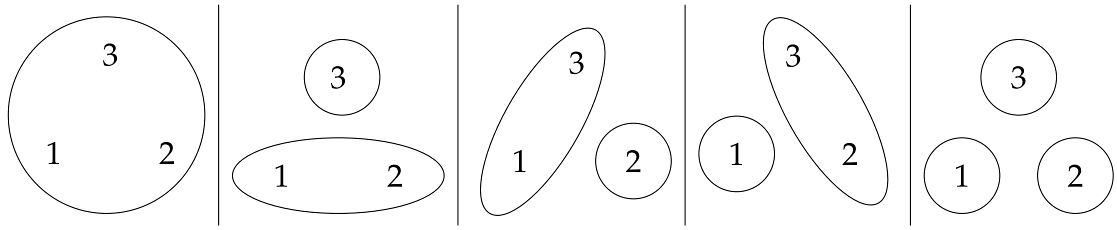
(b) If $\mathcal{F}$ is a set partition of S, then the elements of $\mathcal{F}$ are called the **parts** (or the **blocks**) of $\mathcal{F}$. Keep in mind that they are subsets of S.

(c) If a set partition $\mathcal{F}$ of S has k blocks, then we say that $\mathcal{F}$ is a **set partition of S into k parts**.

Example 3.6.2. Here are all set partitions of the set $[3] = \{1, 2, 3\}$:

$$\{\{1,2,3\}\}, \qquad \{\{1,2\},\{3\}\}, \qquad \{\{1,3\},\{2\}\}, \qquad \{\{2,3\},\{1\}\}, \qquad \{\{1\},\{2\},\{3\}\}.$$

And here are the same set partitions, drawn as pictures (each part of the set partition corresponds to a blob):



The first of these five set partitions has 1 part; the second, third and fourth have 2 parts each; the last has 3 parts.

Proposition 3.6.3. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Let A be an n-element set. Then,

$$(\text{\# of set partitions of } A \text{ into } k \text{ parts}) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}.$$

Proof. Let $X = [k]$. Let us regard the n elements of A as balls, and the k elements of X as boxes. Thus, any surjective $L \to U$ placement from A to X can be viewed as a set partition of A (namely, each box is a part of the set partition, or, more precisely, the balls in this box form a part). To make this rigorous: If p is a surjective map from A to X, then we can construct a set partition $\mathfrak{s}(p)$ of A by setting $\mathfrak{s}(p) = \{A_1, A_2, \ldots, A_k\}$, where we set

$$A_i = \{\text{all balls in box } i \text{ of } p\} = \{a \in A \mid p(a) = i\} \qquad \text{for each } i \in [k].$$

Box-equivalent surjections $p : A \to X$ lead to the **same** set partition $\mathfrak{s}(p)$, and therefore $\mathfrak{s}(p)$ depends not on the surjection $p : A \to X$ but only on its $\overset{\text{box}}{\sim}$-equivalence class. Therefore, we can define $\mathfrak{s}(p)$ not just for a surjection $p : A \to X$ but also for a surjective $L \to U$ placement p (simply by picking an arbitrary surjection p' in p and setting $\mathfrak{s}(p) := \mathfrak{s}(p')$). This gives us a map

$$\mathfrak{s} : \{\text{surjective } L \to U \text{ placements}\} \to \{\text{set partitions of } A \text{ into } k \text{ parts}\},$$
$$p \mapsto \mathfrak{s}(p).$$

It is easy to see that this map $\mathfrak{s}$ is bijective (indeed, the inverse map sends each set partition $\{A_1, A_2, \ldots, A_k\}$ to the surjective $L \to U$ placement that puts all elements of A_1 into box 1, all elements of A_2 into box 2, and so on).[3]

Hence, the bijection principle yields

$$\begin{aligned}
&(\text{\# of set partitions of } A \text{ into } k \text{ parts})\\
&= (\text{\# of surjective } L \to U \text{ placements})\\
&= \left\{ \begin{matrix} |A| \\ |X| \end{matrix} \right\} \qquad \text{(by Proposition 3.5.3)}\\
&= \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad (\text{since } |A| = n \text{ and } |X| = k)\,.
\end{aligned}$$

This proves Proposition 3.6.3. (See [18s-hw3s, Proposition 0.12] for a somewhat different proof.) ☐

Remark 3.6.4. Let $n \in \mathbb{N}$. Let A be an n-element set. Then, a set partition of A cannot have more than n parts (why?). Hence, by the sum rule,

$$\begin{aligned}
&(\text{\# of set partitions of } A)\\
&= \sum_{k=0}^{n} (\text{\# of set partitions of } A \text{ into } k \text{ parts})\\
&= \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{(by Proposition 3.6.3)}\\
&= \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} + \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} + \cdots + \left\{ \begin{matrix} n \\ n \end{matrix} \right\}.
\end{aligned}$$

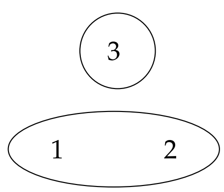
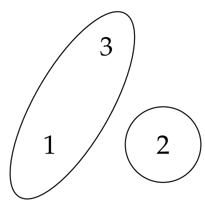
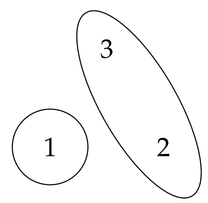
This number is called the ***n*-th Bell number** $B(n)$. For example, $B(3) = 5$, as we see from Example 3.6.2.

There is no explicit formula for $B(n)$, but there is a recursive one:

$$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)\,.$$

This is not hard to prove (nice exercise!). (For a proof, see [Guicha20, Theorem 1.4.3].)

[3]For the sake of illustration, here is a table of values of $\mathfrak{s}$ for $A = [3]$ and $k = 2$:

p	[12] [3]	[13] [2]	[23] [1]
$\mathfrak{s}(p)$			

Remark 3.6.5. Using set partitions, one can generalize the chain rule from analysis to the n-th derivative. To wit, let f and g be two functions from $\mathbb{R}$ to $\mathbb{R}$ that have sufficiently many derivatives. Let $n \in \mathbb{N}$. Then, the n-th derivative $(f \circ g)^{(n)}$ of the composite function $f \circ g$ is given by

$$(f \circ g)^{(n)}(x) = \sum_{\mathcal{F} \text{ is a set partition of } [n]} f^{(|\mathcal{F}|)}(g(x)) \cdot \prod_{B \text{ is a part of } \mathcal{F}} g^{(|B|)}(x).$$

Here, $|\mathcal{F}|$ and $|B|$ are sizes of sets, understood in the usual way (so $|\mathcal{F}|$ is the # of parts of $\mathcal{F}$, and $|B|$ is the # of elements in B). For example, for $n = 3$, this is saying that

$$(f \circ g)^{(3)}(x) = f'(g(x)) \cdot g'''(x) + 3f''(g(x)) \cdot g''(x) \cdot g'(x) + f'''(g(x)) \cdot \left(g'(x)\right)^3.$$

Here, the $f'(g(x)) \cdot g'''(x)$ addend comes from the set partition $\mathcal{F} = \{\{1,2,3\}\}$; the $3f''(g(x)) \cdot g''(x) \cdot g'(x)$ addend comes from the three set partitions into 2 parts (they all contribute equal terms, thus the factor of 3); and the $f'''(g(x)) \cdot (g'(x))^3$ addend comes from the set partition $\mathcal{F} = \{\{1\}, \{2\}, \{3\}\}$.

The above formula for $(f \circ g)^{(n)}$ is known as the **Faa di Bruno formula**. Various proofs can be found in the literature (as can applications, generalizations and equivalent versions). See [Johnso02, §2] for the simplest proof.

3.7. $U \to U$ and integer partitions

3.7.1. Introduction

To complete the twelvefold way, it remains to count $U \to U$ placements.

A $U \to U$ placement can look, for example, as follows:

$$\underline{[\bullet\bullet\bullet]\ [\ ]\ [\bullet]\ [\bullet\bullet]\ [\ ]\ [\ ]\ [\bullet]},$$

with the understanding that the boxes are interchangeable. Thus, we can order the boxes by decreasing number of balls:

$$\underline{[\bullet\bullet\bullet]\ [\bullet\bullet]\ [\bullet]\ [\bullet]\ [\ ]\ [\ ]\ [\ ]}.$$

You can encode this $U \to U$ placement by a sequence of numbers, which say how many balls lie in each box:

$$(3, 2, 1, 1, 0, 0, 0).$$

If the # of boxes is known, we can omit the 0's and just write this as $(3, 2, 1, 1)$. The decreasing order of its entries makes this sequence unique.

3.7.2. Partitions

Let us introduce a name for such sequences:

Definition 3.7.1. A **partition** (or, to be very precise, an **integer partition**) of an integer n is a weakly decreasing list $(a_1, a_2, \ldots, a_k)$ of positive integers whose sum is n (that is, $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ and $a_1 + a_2 + \cdots + a_k = n$).

The entries $a_1, a_2, \ldots, a_k$ of this list are called the **parts** of the partition $(a_1, a_2, \ldots, a_k)$.

If a partition of n has k parts, then we say that it is a **partition of n into k parts**.

Example 3.7.2. The partitions of 5 are

$$(5), \qquad (4,1), \qquad (3,2), \qquad (3,1,1), \qquad (2,2,1),$$
$$(2,1,1,1), \qquad (1,1,1,1,1)$$

For instance, $(2,1,1,1)$ is a partition of 5 into 4 parts.

Remark 3.7.3. A partition of n is the same as a weakly decreasing composition of n.

Definition 3.7.4. Let $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Then, we set

$$p_k(n) := (\text{\# of partitions of } n \text{ into } k \text{ parts}).$$

Example 3.7.5. We have

$$p_0(5) = 0, \qquad p_1(5) = 1, \qquad p_2(5) = 2, \qquad p_3(5) = 2,$$
$$p_4(5) = 1, \qquad p_5(5) = 1, \qquad p_k(5) = 0 \text{ for all } k > 5.$$

There is no explicit formula for $p_k(n)$, but the computation of $p_k(n)$ can still be made fairly painless using the following recursive formula (and base cases):

Proposition 3.7.6. Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

(a) We have $p_k(n) = 0$ when $k < 0$ or $n < 0$.

(b) We have $p_k(n) = 0$ when $k > n$.

(c) We have $p_0(n) = [n = 0]$.

(d) We have $p_1(n) = [n > 0]$.

(e) We have $p_2(n) = \lfloor n/2 \rfloor$ if $n \in \mathbb{N}$.

(f) We have $p_k(n) = p_k(n-k) + p_{k-1}(n-1)$.

Proof. **(a)** A partition has positive entries, so the sum of its entries cannot be negative. Thus, $p_k(n) = 0$ when $n < 0$.

Also, the number of its entries cannot be negative. Thus, $p_k(n) = 0$ when $k < 0$. Proposition 3.7.6 **(a)** is thus proved.

(b) Let $(a_1, a_2, \ldots, a_k)$ be a partition of n into k parts. Then, $a_1, a_2, \ldots, a_k$ are positive integers (by the definition of a partition), and thus are ≥ 1 each. Hence, $\underbrace{a_1}_{\geq 1} + \underbrace{a_2}_{\geq 1} + \cdots + \underbrace{a_k}_{\geq 1} \geq \underbrace{1 + 1 + \cdots + 1}_{k \text{ times}} = k$. However, $a_1 + a_2 + \cdots + a_k = n$ (since $(a_1, a_2, \ldots, a_k)$ is a partition of n). Thus, $n = a_1 + a_2 + \cdots + a_k \geq k$.

Forget that we fixed $(a_1, a_2, \ldots, a_k)$. We thus have shown that if $(a_1, a_2, \ldots, a_k)$ is a partition of n into k parts, then $n \geq k$. Hence, there exists no partition of n into k parts if $n < k$. In other words, $p_k(n) = 0$ when $n < k$. This proves Proposition 3.7.6 **(b)**.

(c) A partition of n into 0 parts must necessarily be the 0-tuple $()$. But the sum of all entries of this 0-tuple is 0 (since an empty sum is 0). Hence, a partition of n into 0 parts exists only if $n = 0$, and in this case it is unique. In other words, the # of partitions of n into 0 parts is $[n = 0]$. In other words, $p_0(n) = [n = 0]$. This proves Proposition 3.7.6 **(c)**.

(d) A partition of n into 1 part must necessarily be the 1-tuple (n) (since the sum of its parts has to be n). But this 1-tuple (n) is a partition only when $n > 0$ (since the entries of a partition have to be positive integers). Thus, a partition of n into 1 part exists only when $n > 0$, and in this case it is unique. In other words, the # of partitions of n into 1 part is $[n > 0]$. In other words, $p_1(n) = [n > 0]$. This proves Proposition 3.7.6 **(d)**.

(e) Assume that $n \in \mathbb{N}$. Then, any partition of n into 2 parts must have the form $(k,\ n-k)$ for some positive integer k (since the sum of its parts must be n). However, $(k,\ n-k)$ is not a partition unless $k \geq n-k$ (because a partition has to be weakly decreasing), i.e., unless $k \geq n/2$. Thus, the partitions of n into 2 parts are

$$(n-1,\ 1), \qquad (n-2,\ 2), \qquad (n-3,\ 3), \qquad \ldots, \qquad (\lceil n/2 \rceil,\ n - \lceil n/2 \rceil).$$

There are clearly $n - \lceil n/2 \rceil = \lfloor n/2 \rfloor$ of these partitions. This proves Proposition 3.7.6 **(e)**.

(f) Let us call a partition

- **red** if 1 is a part of it;
- **green** if 1 is not a part of it.

Recall that a partition must be weakly decreasing (by definition). Thus, the last part of a partition must be its smallest part. Consequently, if 1 is a part of a partition, then the last part of this partition must be 1 (although, of course, some other parts may also be 1). In other words, if a partition is red, then its last part must be 1. Thus, any red partition of n into k parts must necessarily have the form $(a_1, a_2, \ldots, a_{k-1}, 1)$. Removing the last entry from such a red partition yields the $(k-1)$-tuple $(a_1, a_2, \ldots, a_{k-1})$, which is a partition of $n-1$ into $k-1$ parts.

Thus, we obtain a map

$$\begin{aligned} \{\text{red partitions of } n \text{ into } k \text{ parts}\} &\to \{\text{partitions of } n-1 \text{ into } k-1 \text{ parts}\}, \\ (a_1, a_2, \ldots, a_{k-1}, 1) &\mapsto (a_1, a_2, \ldots, a_{k-1}). \end{aligned}$$

Conversely, we have a map

$$\begin{aligned} \{\text{partitions of } n-1 \text{ into } k-1 \text{ parts}\} &\to \{\text{red partitions of } n \text{ into } k \text{ parts}\}, \\ (a_1, a_2, \ldots, a_{k-1}) &\mapsto (a_1, a_2, \ldots, a_{k-1}, 1). \end{aligned}$$

These two maps are mutually inverse and thus are bijections. Hence, the bijection principle yields

$$\begin{aligned} &(\text{\# of red partitions of } n \text{ into } k \text{ parts}) \\ &= (\text{\# of partitions of } n-1 \text{ into } k-1 \text{ parts}) \\ &= p_{k-1}(n-1) \qquad (\text{by definition of } p_{k-1}(n-1)). \end{aligned}$$

On the other hand, if a partition $(a_1, a_2, \ldots, a_k)$ of n is green, then all its parts $a_1, a_2, \ldots, a_k$ are distinct from 1 and therefore larger than 1 (since they are positive integers). Subtracting 1 from each part thus leaves us again with a partition, although this new partition $(a_1 - 1,\ a_2 - 1,\ \ldots,\ a_k - 1)$ will be a partition of $n-k$ (since $(a_1 - 1) + (a_2 - 1) + \cdots + (a_k - 1) = \underbrace{(a_1 + a_2 + \cdots + a_k)}_{=n} - k = n - k$). Hence, we obtain a map

$$\begin{aligned} \{\text{green partitions of } n \text{ into } k \text{ parts}\} &\to \{\text{partitions of } n-k \text{ into } k \text{ parts}\}, \\ (a_1, a_2, \ldots, a_k) &\mapsto (a_1 - 1,\ a_2 - 1,\ \ldots,\ a_k - 1). \end{aligned}$$

Conversely, we have a map

$$\begin{aligned} \{\text{partitions of } n-k \text{ into } k \text{ parts}\} &\to \{\text{green partitions of } n \text{ into } k \text{ parts}\}, \\ (a_1, a_2, \ldots, a_k) &\mapsto (a_1 + 1,\ a_2 + 1,\ \ldots,\ a_k + 1) \end{aligned}$$

(this is well-defined, because $a_1 + 1,\ a_2 + 1,\ \ldots,\ a_k + 1$ will always be distinct from 1 because $a_1, a_2, \ldots, a_k$ are positive). These two maps are mutually inverse and thus are bijections. Hence, the bijection principle yields

$$\begin{aligned} &(\text{\# of green partitions of } n \text{ into } k \text{ parts}) \\ &= (\text{\# of partitions of } n-k \text{ into } k \text{ parts}) \\ &= p_k(n-k) \qquad (\text{by definition of } p_k(n-k)). \end{aligned}$$

Finally, every partition is either red or green (but not both). Hence, by the sum rule,

$$\begin{aligned}
&(\#\text{ of partitions of } n \text{ into } k \text{ parts}) \\
&= \underbrace{(\#\text{ of red partitions of } n \text{ into } k \text{ parts})}_{=p_{k-1}(n-1)} + \underbrace{(\#\text{ of green partitions of } n \text{ into } k \text{ parts})}_{=p_k(n-k)} \\
&= p_{k-1}(n-1) + p_k(n-k).
\end{aligned}$$

Since the left hand side of this equality is $p_k(n)$, we thus have proved Proposition 3.7.6 **(f)**. $\square$

Here is a table of the numbers $p_k(n)$ for small values of k and n:

$p_k(n)$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$k=0$	1	0	0	0	0	0	0	0	0	0
$k=1$	0	1	1	1	1	1	1	1	1	1
$k=2$	0	0	1	1	2	2	3	3	4	4
$k=3$	0	0	0	1	1	2	3	4	5	7
$k=4$	0	0	0	0	1	1	2	3	5	6
$k=5$	0	0	0	0	0	1	1	2	3	5
$k=6$	0	0	0	0	0	0	1	1	2	3
$k=7$	0	0	0	0	0	0	0	1	1	2
$k=8$	0	0	0	0	0	0	0	0	1	1
$k=9$	0	0	0	0	0	0	0	0	0	1

You may spot a few more properties of $p_k(n)$ in this table, such as the following:

Exercise 1. Show that $p_{n-1}(n) = 1$ for each $n \geq 2$.

Exercise 2. More generally, show that $p_{n-k}(n) = p_k(2k)$ for each $k \in \mathbb{N}$ and each $n \geq 2k$.

Exercise 3 ((harder))**.** Let $n \in \mathbb{N}$. Show that $p_3(n) = \operatorname{round} \dfrac{n^2}{12}$, where $\operatorname{round} x$ denotes the closest integer to x.

See Sequence A008284 in the OEIS for more about the numbers $p_k(n)$.

3.7.3. Counting $U \to U$ placements

Now, back to the boxes and the balls. We can at last count $U \to U$ placements:

Proposition 3.7.7. We have

$$(\text{\# of surjective } U \to U \text{ placements } A \to X) = p_{|X|}(|A|).$$

Proof. In a surjective $U \to U$ placement, any given box has a positive # of balls. However, the boxes are unlabelled, so we don't know which box is the 1-st box, which is the 2-nd, and so on. However, we can order the boxes by decreasing # of balls, and then there is a well-defined "1-st box", a well-defined "2-nd box", etc..

Thus, we can encode a surjective $U \to U$ placement $A \to X$ by the $|X|$-tuple $\left(a_1, a_2, \ldots, a_{|X|}\right)$, where

$$a_i = (\text{\# of balls in the } i\text{-th box})$$

(where the boxes are ordered by decreasing # of balls). This $|X|$-tuple $\left(a_1, a_2, \ldots, a_{|X|}\right)$ is weakly decreasing (because of how we ordered the boxes), and its entries are positive integers (since our placement is surjective), and the sum of its entries is $|A|$ (since we have a total of $|A|$ many balls). Thus, this $|X|$-tuple is a partition of $|A|$ into $|X|$ parts.

This allows us to define a map

$$\{\text{surjective } U \to U \text{ placements } A \to X\} \to \{\text{partitions of } |A| \text{ into } |X| \text{ parts}\},$$

which sends a placement to the $|X|$-tuple $\left(a_1, a_2, \ldots, a_{|X|}\right)$ we just defined. It is easy to see that this map is bijective. Thus, by the bijection principle,

$$\begin{aligned}
&(\text{\# of surjective } U \to U \text{ placements } A \to X) \\
&= (\text{\# of partitions of } |A| \text{ into } |X| \text{ parts}) = p_{|X|}(|A|).
\end{aligned}$$

□

Proposition 3.7.8. We have

$$(\text{\# of injective } U \to U \text{ placements } A \to X) = [|A| \leq |X|].$$

Proof. Easy exercise. □

Proposition 3.7.9. We have

$$\begin{aligned}
&(\text{\# of } U \to U \text{ placements } A \to X) \\
&= p_0(|A|) + p_1(|A|) + \cdots + p_{|X|}(|A|) \\
&= \sum_{k=0}^{|X|} p_k(|A|).
\end{aligned}$$

Proof sketch. As in the proof of Proposition 3.5.4, we break the sum up according to the # of nonempty boxes. (Details left to the reader.) □

Exercise 4. Prove that the # in Proposition 3.7.9 is also equal to $p_{|X|}(|X| + |A|)$.

We can now fill in the twelvefold way table completely:

	arbitrary	injective	surjective
$L \to L$	$\|X\|^{\|A\|}$	$\|X\|^{\underline{\|A\|}}$	$\text{sur}(\|X\|, \|A\|)$
$U \to L$	$\binom{\|A\| + \|X\| - 1}{\|A\|}$	$\binom{\|X\|}{\|A\|}$	$\binom{\|A\| - 1}{\|A\| - \|X\|}$
$L \to U$	$\sum_{k=0}^{\|X\|} \left\{ \begin{matrix} \|A\| \\ k \end{matrix} \right\}$	$[\|A\| \leq \|X\|]$	$\left\{ \begin{matrix} \|A\| \\ \|X\| \end{matrix} \right\}$
$U \to U$	$\sum_{k=0}^{\|X\|} p_k(\|A\|)$	$[\|A\| \leq \|X\|]$	$p_{\|X\|}(\|A\|)$

.

3.7.4. More about partition numbers

The numbers $p_k(n)$ have many interesting properties. Yet more mysterious are the numbers

$$p(n) := (\text{\# of all partitions of } n) = \sum_{k=0}^{n} p_k(n).$$

These numbers $p(n)$ are called the **partition numbers**; here is a little table of some of them:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135

.

One of their most famous properties is the following recursion formula found by Euler:

Theorem 3.7.10 (Euler's recursion for the partition numbers)**.** For each $n > 0$, we have

$$p(n) = \sum_{\substack{k \in \mathbb{Z}; \\ k \neq 0}} (-1)^{k-1} p(n - w_k). \qquad (1)$$

Here, the numbers w_k are the so-called **pentagonal numbers**, defined by

$$w_k := \frac{(3k-1)k}{2} \qquad \text{for each } k \in \mathbb{Z}.$$

Here is a little table of the smallest few of them:

k	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
w_k	77	57	40	26	15	7	2	0	1	5	12	22	35	51	70

.

Note that they decrease for $k \leq 0$ and increase for $k \geq 0$. For each given n, only finitely many addends on the right hand side of (1) are nonzero (because if $|k|$ is sufficiently large, we have $w_k > n$ and thus $p(n - w_k) = p(\text{something negative}) = 0$), and therefore the infinite sum in (1) is well-defined. "Explicitly", (1) takes the form

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) \pm \cdots.$$

For a proof of Theorem 3.7.10, see a graduate course on combinatorics (e.g., [21s, Corollary 4.2.3]). Readers familiar with power series should take a look at the Wikipedia page for Euler's Pentagonal Number Theorem, of which Theorem 3.7.10 is an easy corollary. (Self-contained proofs of Euler's Pentagonal Number Theorem can also be found in [Bell06, §3], [Koch16, §10], [Zabroc03] or [Sills12, §2.6–§2.7].)

Interested in more? A book-length introduction to the vast and deep theory of partitions is [AndEri04] (although it is not always fully rigorous, but it gives a good idea of how results in this subject look like). My notes [21s, Chapter 4] give an introduction as well (more detailed but also much more modest in coverage). Most advanced courses in enumerative (or algebraic) combinatorics contain at least some material on partitions.

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