Math 222 Fall 2022, Lecture 24: The twelvefold way

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

0.1. Solving the counting puzzle from Lecture 23

In Lecture 23, we posed the following problem:

Exercise 1. Given n persons ($n > 0$) and k tasks ($k > 0$).

(a) What is the # of ways to assign a task to each person such that each task has at least 1 person working on it?

(b) What if we additionally want to choose a leader for each task? (The leader should be chosen from the people working on that task.)

(c) What if, instead of choosing leaders, we want to choose a vertical hierarchy for each task? (A vertical hierarchy means a ranking of all people working on this task, from highest to lowest; ties are not allowed.)

Example: Assume that $n = 8$ and $k = 3$. Let our 8 people be $1, 2, 3, 4, 5, 6, 7, 8$, and our 3 tasks be A, B, C.

(a) One option is

task	people working on it
A	$1, 2, 5$
B	3
C	$4, 6, 7, 8$

.

(b) One option is

task	people working on it
A	$1, 2, 5$ with leader 2
B	3 with leader 3
C	$4, 6, 7, 8$ with leader 7

.

(c) One option is

task	people working on it
A	$1 > 5 > 2$
B	3
C	$7 > 8 > 4 > 6$

(where the "$>$" sign means "is ranked above").

Let us now answer it.

Solution to Exercise 1. **(a)** The answer is $\operatorname{sur}(n,k)$.

Proof. Choosing such an assignment is tantamount to choosing a surjection from {people} to {tasks}.

(b) The answer is $n^{\underline{k}} \cdot k^{n-k}$.

Proof. First, choose a leader for each task. There are $n^{\underline{k}}$ options for this (since the leaders have to be distinct, so we are choosing an injective map from {tasks} to {people}). After the leaders have been chosen, each of the remaining $n-k$ people joins one of the k leaders. There are k^{n-k} options for this.

(c) The answer is $n! \cdot \dbinom{n-1}{k-1}$.

Proof. WLOG, we assume that the n people are called $1, 2, \ldots, n$, and the k tasks are called $1, 2, \ldots, k$.

First, we choose the k-tuple $(a_1, a_2, \ldots, a_k)$, where a_i is the # of people to work on task i. This k-tuple is a composition of n into k parts (since its entries $a_1, a_2, \ldots, a_k$ have to be positive integers and to sum up to n). Thus, there are $\dbinom{n-1}{k-1}$ options for it (by Theorem 2.8.1 in Lecture 21). We shall refer to this composition as the **headcount composition**.

Next, we choose a permutation σ of $[n]$, and we put all the n people into a single vertical hierarchy:

$$\sigma(1) > \sigma(2) > \sigma(3) > \cdots > \sigma(n)$$

(where the symbol "$>$" means "is ranked above", not "larger as a number"). This hierarchy will be called the **total hierarchy**.

Now, we assign the top a_1 people of this total hierarchy to task 1, the next-highest a_2 people to task 2, and so on. This way, we have chosen both an assignment of tasks and a hierarchy within each task. For example, if $n = 8$ and $k = 3$ and $(a_1, a_2, a_3) = (3, 1, 4)$, and if σ is the permutation of $[8]$ whose one-line notation is $(1, 5, 2, 3, 7, 8, 4, 6)$ (so that the total hierarchy is $1 > 5 > 2 > 3 > 7 > 8 > 4 > 6$), then we get the following result:

task	people working on it
A	$1 > 5 > 2$
B	3
C	$7 > 8 > 4 > 6$

(which is precisely the example given in the exercise).

Thus, by choosing a composition $(a_1, a_2, \ldots, a_k)$ of n into k parts and a permutation σ of $[n]$, we have obtained an assignment of tasks and a hierarchy within each task. Note that each choice of an assignment of tasks and a hierarchy within each task can be constructed uniquely in this way (i.e., it is obtained from a unique composition $(a_1, a_2, \ldots, a_k)$ and a unique permutation σ of $[n]$). There are $\dbinom{n-1}{k-1}$ options for the composition $(a_1, a_2, \ldots, a_k)$, and $n!$ options for the permutation σ. Thus, the total # of possibilities is $\dbinom{n-1}{k-1} \cdot n! = n! \cdot \dbinom{n-1}{k-1}$. □

3. The twelvefold way

Now, back to our balls and boxes. Recall from Lecture 23:

- We are counting ways to place balls into boxes.

  The balls are the elements of a given set A.

  The boxes are the elements of a given set X.

  We have defined an $L \to L$ **placement** to be a map from A to X. This is understood to be a way to place the balls in the boxes.

  We have counted arbitrary, surjective and injective $L \to L$ placements.

- We still want to count:
  - $L \to U$ placements (i.e., placements where the boxes are unlabelled, aka indistinguishable),
  - $U \to L$ placements (i.e., placements where the balls are unlabelled),
  - $U \to U$ placements (i.e., placements where everything is unlabelled).

3.3. Relations and equivalence classes

3.3.1. What we need

First, we need to define what "unlabelled" means. The rigorous way to do so is via the concept of **equivalence classes**. The idea behind this is to collect certain $L \to L$ placements into sets (these sets are the equivalence classes) and to count not the placements themselves but rather the sets.

Specifically:

- The $L \to U$ placements are **sets** of $L \to L$ placements, where we throw two $L \to L$ placements into the same set if they can be obtained from one another by permuting boxes.

- The $U \to L$ placements are **sets** of $L \to L$ placements, where we throw two $L \to L$ placements into the same set if they can be obtained from one another by permuting balls.
- The $U \to U$ placements are **sets** of $L \to L$ placements, where we throw two $L \to L$ placements into the same set if they can be obtained from one another by permuting boxes and permuting balls.

This "throwing together" procedure is easiest to formalize by defining **relations**.

3.3.2. Relations

What is a relation? For instance, divisibility ("x divides y") is a relation on the set $\mathbb{Z}$, since two integers x and y can either satisfy or not satisfy "x divides y". This relation is usually denoted by $\mid$. Other famous relations on the set $\mathbb{Z}$ are $\leq$, $\geq$, $<$, $>$, $=$, $\neq$ and $\vdots$. (The relation $\vdots$ is defined as follows: $x \vdots y$ if and only if $y \mid x$.) Many more relations have no standard notation for them, but can be easily defined. For instance, the relation "x and y are consecutive integers" (that is, "$y = x + 1$ or $y = x - 1$") on $\mathbb{Z}$ is an important relation in the study of lacunar sets.

Here is the general definition of a relation:

Definition 3.3.1. Let S be a set.

A **binary relation** (or, for short, **relation**) on S is formally defined as a subset of $S \times S$. If R is a binary relation on S, then the statement "$(x, y) \in R$" is commonly written as "$x \ R \ y$".

Informally, a binary relation on S means a statement "$x \ R \ y$" defined for every pair of elements $(x, y) \in S \times S$. For each pair $(x, y) \in S \times S$, this statement $x \ R \ y$ is either true or false.

The informal and the formal definitions of a binary relation are equivalent, because:

- Given a statement "$x \ R \ y$" defined for every pair $(x, y) \in S \times S$, we can encode it as a subset of $S \times S$, namely as the subset

$$\{(x, y) \in S \times S \ \mid \ x \ R \ y\}.$$

- Conversely, every subset T of $S \times S$ can be decoded into a statement defined for every pair $(x, y) \in S \times S$ (namely, the statement "$(x, y) \in T$").

Example 3.3.2. Consider the relation $\mid$ ("divides"), but not on the entire set $\mathbb{Z}$ but rather on its subset $[6] = \{1,2,3,4,5,6\}$. Then, formally speaking, this relation is the following subset of $[6] \times [6]$:

$$\{(1,1)\,,\ (1,2)\,,\ (1,3)\,,\ (1,4)\,,\ (1,5)\,,\ (1,6)\,,\ (2,2)\,,\ (2,4)\,,\ (2,6)\,,\\ (3,3)\,,\ (3,6)\,,\ (4,4)\,,\ (5,5)\,,\ (6,6)\}.$$

It consists of all pairs $(x,y) \in [6] \times [6]$ such that x divides y.

3.3.3. Box-, ball- and box-ball-equivalence

Now, let us apply this to boxes and balls:

Definition 3.3.3. Consider the set X^A of all $L \to L$ placements (i.e., of all maps from A to X). On this set, we define three relations

$$\overset{\text{box}}{\sim}, \qquad \overset{\text{ball}}{\sim}, \qquad \overset{\text{box}}{\underset{\text{ball}}{\sim}},$$

which we will call **box-equivalence**, **ball-equivalence** and **box-ball-equivalence** (respectively). These relations are defined as follows:

- Two $L \to L$ placements $f, g : A \to X$ are said to be **box-equivalent** (and we write $f \overset{\text{box}}{\sim} g$) if they can be obtained from one another by permuting the boxes (i.e., if there exists a permutation ξ of X such that $f = \xi \circ g$).
- Two $L \to L$ placements $f, g : A \to X$ are said to be **ball-equivalent** (and we write $f \overset{\text{ball}}{\sim} g$) if they can be obtained from one another by permuting the balls (i.e., if there exists a permutation α of A such that $f = g \circ \alpha$).
- Two $L \to L$ placements $f, g : A \to X$ are said to be **box-ball-equivalent** (and we write $f \overset{\text{box}}{\underset{\text{ball}}{\sim}} g$) if they can be obtained from one another by permuting the balls and the boxes (i.e., if there exist a permutation α of A and a permutation ξ of X such that $f = \xi \circ g \circ \alpha$).

Example 3.3.4. We shall use the notations from §3.1 (in Lecture 23) to write $L \to L$ placements. Let $A = [6]$ and $X = [3]$ (so that the balls are $1, 2, \ldots, 6$, and the boxes are $1, 2, 3$). Consider the two $L \to L$ placements

$$f = [14]\ \ [235]\ \ [6] \qquad \text{and} \qquad g = [125]\ \ [36]\ \ [4]\,.$$

These two $L \to L$ placements f and g are not box-equivalent (because no matter how we permute the boxes in f, the two balls 1 and 4 will stay together in the same box, but these two balls lie in different boxes of g). They are not ball-equivalent either (because no matter how we permute the balls in f, the first box will have 2 balls, but in g it has 3 balls). However, they are box-ball-equivalent, because if we start with f, then swap the first two boxes and furthermore appropriately permute the balls (replacing balls $1, 2, 3, 4, 5, 6$ by $3, 1, 2, 6, 5, 4$, respectively), then we obtain g.

3.3.4. Equivalence classes

The three relations introduced in Definition 3.3.3 are there to tell us which of our $L \to L$ placements we should throw into the same set when we count $L \to U$ placements, $U \to L$ placements and $U \to U$ placements. (For instance, in order to count $L \to U$ placements, we need to throw any two box-equivalent placements into the same set.)

The resulting sets will be called the **equivalence classes**. Here is how these classes are defined in general:

Definition 3.3.5. Let $\sim$ be a relation on a set S.

(a) For each $a \in S$, we define a subset $[a]_\sim$ of S by

$$[a]_\sim = \{b \in S \mid b \sim a\}.$$

This subset $[a]_\sim$ is called the **equivalence class** of a (for the relation $\sim$), or the **$\sim$-equivalence class** of a.

(b) The **equivalence classes of** $\sim$ are defined to be the sets $[a]_\sim$ for all $a \in S$. They are also called the **$\sim$-equivalence classes**.

Example 3.3.6. Let S be the set $\mathbb{Z}$, and let $\sim$ be the relation "divides" (i.e., we let $b \sim a$ hold if and only if b divides a). Then, for any integer a, the equivalence class $[a]_\sim$ is the set of all divisors of a. For instance,

$$[6]_\sim = \{1, 2, 3, 6, -1, -2, -3, -6\}.$$

Definition 3.3.5 is slightly nonstandard. Usually, one defines equivalence classes only for equivalence relations, not for arbitrary relations. I have made it more general in order to get to the point faster. Now, we can define the placements we want:

Definition 3.3.7. **(a)** An $L \to U$ **placement** (from A to X) shall mean a $\overset{\text{box}}{\sim}$-equivalence class.

(b) A $U \to L$ **placement** (from A to X) shall mean a $\overset{\text{ball}}{\sim}$-equivalence class.

(c) A $U \to U$ **placement** (from A to X) shall mean a $\overset{\text{box}}{\underset{\text{ball}}{\sim}}$-equivalence class.

(d) Let ϕ be an $L \to U$ or $U \to L$ or $U \to U$ placement. Then, ϕ is a set of maps from A to X. We say that ϕ is **injective** if all of these maps in ϕ are injective. We say that ϕ is **surjective** if all of these maps in ϕ are surjective.

Example 3.3.8. Let $X = [2]$ and $A = [3]$. As we found in Lecture 23, there are eight $L \to L$ placements $A \to X$, namely

$$\begin{gathered}
[123]\ [\ ], \\
[12]\ [3], \\
[13]\ [2], \\
[23]\ [1], \\
[1]\ [23], \\
[2]\ [13], \\
[3]\ [12], \\
[\ ]\ [123].
\end{gathered}$$

Now let us see what the $L \to U$, $U \to L$ and $U \to U$ placements are, defined as equivalence classes:

(a) The $L \to U$ placements are the four $\overset{\text{box}}{\sim}$-equivalence classes

$$\begin{aligned}
&\{[123]\ [\ ], \quad [\ ]\ [123]\}, \\
&\{[12]\ [3], \quad [3]\ [12]\}, \\
&\{[13]\ [2], \quad [2]\ [13]\}, \\
&\{[23]\ [1], \quad [1]\ [23]\}.
\end{aligned}$$

In the notations of Lecture 23, they are denoted by

$$\begin{gathered}
\underline{[123]\ [\ ]}, \\
\underline{[12]\ [3]}, \\
\underline{[13]\ [2]}, \\
\underline{[23]\ [1]},
\end{gathered}$$

respectively.

(b) The $U \to L$ placements are the four $\overset{\text{ball}}{\sim}$-equivalence classes

$$\begin{aligned}
&\{[123]\ [\ ]\}, \\
&\{[12]\ [3], \quad [13]\ [2], \quad [23]\ [1]\}, \\
&\{[1]\ [23], \quad [2]\ [13], \quad [3]\ [12]\}, \\
&\{[\ ]\ [123]\}.
\end{aligned}$$

In the notations of Lecture 23, they are denoted by

$$\begin{gathered}[\bullet\bullet\bullet]\ [\ ],\\ [\bullet\bullet]\ [\bullet],\\ [\bullet]\ [\bullet\bullet],\\ [\ ]\ [\bullet\bullet\bullet],\end{gathered}$$

respectively.

(c) The $U \to U$ placements are the two $\underset{\text{ball}}{\overset{\text{box}}{\sim}}$-equivalence classes

$$\{[123]\ [\ ],\quad [\ ]\ [123]\},$$
$$\{[12]\ [3],\quad [13]\ [2],\quad [23]\ [1],\quad [1]\ [23],\quad [2]\ [13],\quad [3]\ [12]\}.$$

In the notations of Lecture 23, they are denoted by

$$\begin{gathered}\underline{[\bullet\bullet\bullet]\ [\ ]},\\ \underline{[\bullet\bullet]\ [\bullet]},\end{gathered}$$

respectively.

Now we know precisely what we want to count!

3.3.5. Equivalence relations

Before we get to the actual counting, a few words about equivalence classes are in order. As I said, we defined them rather generally. In this generality, there is not much to say about these equivalence classes. They can overlap, be empty, contain one another, etc..

However, they behave much better if $\sim$ is an **equivalence relation**. Let's briefly define what this means:

Definition 3.3.9. Let $\sim$ be a relation on a set S. We say that $\sim$ is an **equivalence relation** if it satisfies the following three axioms:

- **Reflexivity:** For any $a \in S$, we have $a \sim a$.
- **Symmetry:** For any $a, b \in S$, if $a \sim b$, then $b \sim a$.
- **Transitivity:** For any $a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

The relation "divides" on $\mathbb{Z}$ is not an equivalence relation (since it fails symmetry). The relation "is less than 1 mile away from" (e.g., on the set of all buildings in Philadelphia) is not an equivalence relation either (since it fails transitivity). However, the relation "equals" (on an arbitrary set) is an equivalence relation, as is the relation "parallel" (on the set of all straight lines in the Euclidean plane), as is the relation "is congruent modulo 5" (on the set of all integers). See §3.3.2 in the 2019 notes for many more examples of equivalence relations.

The example we care about the most is the following:

Proposition 3.3.10. All three relations

$$\overset{\text{box}}{\sim}, \qquad \overset{\text{ball}}{\sim}, \qquad \underset{\text{ball}}{\overset{\text{box}}{\sim}}$$

on the set X^A are equivalence relations.

Proof. See Exercise 3.3.3 in the 2019 notes. The proof is a straightforward application of the following three simple facts:

- If α and β are two permutations of the same set, then $\alpha \circ \beta$ is also a permutation of this set.
- If α is a permutation of a set, then α^{-1} is also a permutation of this set.
- The identity map id_X of a set X is always a permutation of X.

□

Why are we singling out equivalence relations? The reason is that if $\sim$ is an equivalence relation, then its equivalence classes are particularly well-behaved:

Theorem 3.3.11. Let $\sim$ be an equivalence relation on a set S. Then, each element s of S belongs to **exactly one** $\sim$-equivalence class, namely to its own $\sim$-equivalence class $[s]_\sim$.

(Note that the word "exactly one" means that if s belongs to two $\sim$-equivalence classes $[a]_\sim$ and $[b]_\sim$, then these two classes $[a]_\sim$ and $[b]_\sim$ are identical, even though we don't necessarily have $a = b$.)

3.4. $U \to L$ placements

Time to count!

Recall that the $U \to L$ placements are the $\overset{\text{ball}}{\sim}$-equivalence classes. We will write them using the $\bullet$-notation (as explained in Example §3.1 in Lecture 23). For example, $[\bullet\,\bullet]\ [\bullet]$ means the $U \to L$ placement that puts two balls into box 1 and one ball into box 2.

Proposition 3.4.1. We have

$$\begin{aligned}
&(\text{\# of } U \to L \text{ placements } A \to X) \\
&= \left(\text{\# of } \left(x_1, x_2, \ldots, x_{|X|}\right) \in \mathbb{N}^{|X|} \text{ satisfying } x_1 + x_2 + \cdots + x_{|X|} = |A|\right) \\
&= \binom{|A| + |X| - 1}{|A|}.
\end{aligned}$$

Proof. The second equality sign follows from Theorem 2.8.5 in Lecture 21. It remains to prove the first equality sign.

WLOG assume that the boxes are called $1, 2, \ldots, |X|$. (This is a harmless assumption, because of the isomorphism principle.)

Now, define the **occupancy list** of an ($L \to L$ or $U \to L$) placement to be the $|X|$-tuple

$$(\text{\# of balls in box } 1, \ \ \text{\# of balls in box } 2, \ \ \ldots, \ \ \text{\# of balls in box } |X|) \in \mathbb{N}^X.$$

So the occupancy list just remembers how many balls there are in each single box. This is clearly well-defined, since ball-equivalent $L \to L$ placements have the same occupancy list (after all, swapping two balls doesn't change the # of balls in any given box).

Now, consider the map

$$\{U \to L \text{ placements}\} \to \left\{\left(x_1, x_2, \ldots, x_{|X|}\right) \in \mathbb{N}^{|X|} \ \mid \ x_1 + x_2 + \cdots + x_{|X|} = |A|\right\}$$

that sends each $U \to L$ placement to its occupancy list. This map is a bijection, because

- it is injective (any two $L \to L$ placements that have the same occupancy list must be ball-equivalent, and thus constitute the same $U \to L$ placement), and
- it is surjective (any $|X|$-tuple $\left(x_1, x_2, \ldots, x_{|X|}\right) \in \mathbb{N}^{|X|}$ satisfying $x_1 + x_2 + \cdots + x_{|X|} = |A|$ is the occupancy list of a $U \to L$ placement, since we can simply put x_1 balls into box 1, put x_2 balls into box 2, and so on).

Hence, the bijection principle yields

$$\begin{aligned}
&(\text{\# of } U \to L \text{ placements } A \to X) \\
&= \left(\text{\# of } \left(x_1, x_2, \ldots, x_{|X|}\right) \in \mathbb{N}^{|X|} \text{ satisfying } x_1 + x_2 + \cdots + x_{|X|} = |A|\right),
\end{aligned}$$

and this proves the first equality sign. Proposition 3.4.1 is thus proven. $\square$

Proposition 3.4.2. Let $\mathbb{P} = \{1, 2, 3, \ldots\}$. Then,

$$\begin{aligned}
&(\text{\# of surjective } U \to L \text{ placements } A \to X) \\
&= \left(\text{\# of } \left(x_1, x_2, \ldots, x_{|X|}\right) \in \mathbb{P}^{|X|} \text{ satisfying } x_1 + x_2 + \cdots + x_{|X|} = |A|\right) \\
&= \binom{|A| - 1}{|A| - |X|}.
\end{aligned}$$

Proof. The first equality sign is proved similarly as in Proposition 3.4.1, but now the occupancy list consists of **positive** integers (since a surjective $U \to L$ placement must not leave any box empty). The second equality sign follows from Theorem 2.8.1 in Lecture 21. □

Proposition 3.4.3. We have

$$\begin{aligned}
&(\text{\# of injective } U \to L \text{ placements } A \to X) \\
&= \left(\text{\# of } \left(x_1, x_2, \ldots, x_{|X|}\right) \in \{0, 1\}^{|X|} \text{ satisfying } x_1 + x_2 + \cdots + x_{|X|} = |A|\right) \\
&= \binom{|X|}{|A|}.
\end{aligned}$$

Proof. The first equality sign is proved similarly as in Proposition 3.4.1, but now the occupancy list consists of 0's and 1's (since a box in an injective $U \to L$ placement can only contain 0 or 1 balls). The second equality sign follows from Theorem 2.8.4 in Lecture 21. □

We can now fill in the first two rows of our twelvefold way table:

	arbitrary	injective	surjective
$L \to L$	$\|X\|^{\|A\|}$	$\|X\|^{\underline{\|A\|}}$	$\operatorname{sur}(\|X\|, \|A\|)$
$U \to L$	$\binom{\|A\| + \|X\| - 1}{\|A\|}$	$\binom{\|X\|}{\|A\|}$	$\binom{\|A\| - 1}{\|A\| - \|X\|}$
$L \to U$			
$U \to U$			

.

3.5. $L \to U$ placements

Now, let us move on to the $L \to U$ placements. These, as we recall, are the $\overset{\text{box}}{\sim}$-equivalence classes. We again want to count arbitrary, surjective and injective $L \to U$ placements. The injective ones are the easiest to count:

Proposition 3.5.1. We have

$$\begin{aligned}&(\text{\# of injective } L \to U \text{ placements } A \to X) \\ &= [|A| \leq |X|].\end{aligned}$$

Proof. We WLOG assume that the balls are called $1, 2, \ldots, |A|$, and that the boxes are called $1, 2, \ldots, |X|$. Thus, $A = \{1, 2, \ldots, |A|\} = [|A|]$ and $X = \{1, 2, \ldots, |X|\} = [|X|]$.

If $|A| > |X|$, then there are no injective maps from A to X (by the Pigeonhole Principle for Injections) and thus also no injective $L \to U$ placements (since these are equivalence classes of injective maps from A to X). Thus, the claim of Proposition 3.5.1 boils down to $0 = 0$ in this case.

It remains to handle the case when $|A| \leq |X|$. In this case, there is at least one injective $L \to L$ placement $A \to X$, namely the $L \to L$ placement

$$[1]\ [2]\ \cdots\ [|A|]\ [\ ]\ [\ ]\ \cdots\ [\ ].$$

Let us denote this $L \to L$ placement by f.

Now, every injective $L \to L$ placement is box-equivalent to f [1]. Hence, there is only one box-equivalence class consisting of injective maps, namely $[f]_{\underset{\text{box}}{\sim}}$. In other words, there is only one injective $L \to U$ placement $A \to X$, namely $[f]_{\underset{\text{box}}{\sim}}$.

Therefore, (# of injective $L \to U$ placements $A \to X$) $= 1 = [|A| \leq |X|]$. This proves Proposition 3.5.1. □

Next time, we will count surjective and general $L \to U$ placements. This is far less easy.

[1] *Proof.* Let g be an injective $L \to L$ placement. We must prove that g is box-equivalent to f.

Note that $A = \{1, 2, \ldots, |A|\} = [|A|]$ and $X = \{1, 2, \ldots, |X|\} = [|X|]$.

The placement g must have a box (namely, $g(1)$) that contains the ball 1; a box (namely, $g(2)$) that contains the ball 2; and so on. Consider these $|A|$ boxes $g(1), g(2), \ldots, g(|A|)$. All these boxes are distinct (since g is injective, so that no two balls are in the same box). Thus, we can find a permutation σ of $X = [|X|]$ that sends $1, 2, \ldots, |A|$ to $g(1), g(2), \ldots, g(|A|)$. (Indeed, we can construct such a permutation σ by sending the first $|A|$ positive integers $1, 2, \ldots, |A|$ to $g(1), g(2), \ldots, g(|A|)$, while sending the next $|X| - |A|$ positive integers $|A| + 1, |A| + 2, \ldots, |X|$ to the remaining $|X| - |A|$ elements of X.)

This permutation σ then has the property that $g = \sigma \circ f$ (check this!). Hence, f and g are box-equivalent (by the definition of box-equivalence). Qed.