Math 222 Fall 2022, Lecture 22: Binomial coefficients

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

2. Binomial coefficients (cont'd)

2.9. Multisubsets (cont'd)

Recall the following definitions from Lecture 21:

- A **multisubset** of a set T is defined to be a map $f : T \to \mathbb{N}$ such that only finitely many $t \in T$ satisfy $f(t) \neq 0$. We regard such a multisubset f as a "set with multiplicities", which contains each $t \in T$ a total of $f(t)$ many times.
- The **size** of this multisubset f is defined to be the sum $\sum_{t \in T} f(t)$. This size is always an element of $\mathbb{N}$.
- We introduced the notation $\{a_1, a_2, \ldots, a_k\}_{\text{multi}}$, where $a_1, a_2, \ldots, a_k$ are some elements of a given set T. This notation stands for a certain multisubset of T having size k; namely, this multisubset is formally defined as the function $f : T \to \mathbb{N}$, where $f(t)$ is the # of times that t appears in the tuple $(a_1, a_2, \ldots, a_k)$.

2.9.2. Counting

For a given n-element set T and a given real k, we know (from Lecture 6, Theorem 1.3.10) that

$$(\text{\# of subsets } T \text{ having size } k) = \binom{n}{k}.$$

A similar formula exists for counting multisubsets with size k:

Corollary 2.9.3. Let $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Let T be an n-element set. Then,

$$(\text{\# of multisubsets of } T \text{ having size } k) = \binom{k+n-1}{k}.$$

Example 2.9.4. The multisubsets of $[3]$ having size 2 are

$$\{1,1\}_{\text{multi}}, \quad \{1,2\}_{\text{multi}}, \quad \{1,3\}_{\text{multi}}, \quad \{2,2\}_{\text{multi}}, \quad \{2,3\}_{\text{multi}}, \quad \{3,3\}_{\text{multi}}.$$

Thus, there are 6 of them. On the other hand, $\dbinom{2+3-1}{2} = 6$. Thus, Corollary 2.9.3 holds for $n = 3$ and $k = 2$.

Proof of Corollary 2.9.3. We WLOG assume that $k \in \mathbb{N}$ (otherwise, we are proving $0 = 0$).

Theorem 2.8.5 in Lecture 21 (applied to k and n instead of n and k) shows that

$$\begin{aligned}&(\text{\# of } (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \text{ satisfying } x_1 + x_2 + \cdots + x_n = k)\\ &= \dbinom{k+n-1}{k}.\end{aligned}$$

So it remains to prove that

$$\begin{aligned}&(\text{\# of multisubsets of } T \text{ having size } k)\\ &= (\text{\# of } (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \text{ satisfying } x_1 + x_2 + \cdots + x_n = k). \qquad (1)\end{aligned}$$

We shall obtain this from the bijection principle.

Label the n elements of T as $t_1, t_2, \ldots, t_n$. Then, each multisubset of T is a function $f : T \to \mathbb{N}$, and therefore is uniquely determined by its n values $f(t_1), f(t_2), \ldots, f(t_n)$. Moreover, the size of this multisubset f is $\sum_{t \in T} f(t) = f(t_1) + f(t_2) + \cdots + f(t_n)$. Hence, the multisubset f has size k if and only if $f(t_1) + f(t_2) + \cdots + f(t_n) = k$.

Therefore, the multisubsets of T having size k are the functions $f : T \to \mathbb{N}$ that satisfy $f(t_1) + f(t_2) + \cdots + f(t_n) = k$. We thus obtain a bijection

$$\begin{aligned}&\text{from } \{\text{multisubsets of } T \text{ having size } k\}\\ &\text{to } \{(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \mid x_1 + x_2 + \cdots + x_n = k\},\end{aligned}$$

which sends each multisubset f to the n-tuple $(f(t_1), f(t_2), \ldots, f(t_n))$. The bijection principle therefore yields the equality (1), and this completes our proof.

(For more details, see the proof of Corollary 2.11.3 in the 2019 notes, but keep in mind that the numbers n and k there are the k and n here.) $\square$

Note that the $\dbinom{k+n-1}{k}$ in Corollary 2.9.3 can also be rewritten as $(-1)^k \dbinom{-n}{k}$ (by the upper negation formula). This suggests (if one is sufficiently quick at jumping to conclusions) that somehow, the passage from sets to multisets somehow introduces minus signs into formulas. This is not as far-fetched as it might seem; similar behavior has been observed in more complicated combinatorial questions! Such results are commonly known as **combinatorial reciprocities**, and I refer to [BecSan18] for a deeper study of many of them.

2.9.3. An application to lacunar subsets

We shall now see an application of multisubsets to counting sets (not multisets!).

First, we need a few features of multisubsets. We begin by recalling the following basic property of sets:

Proposition 2.9.5. Let $m \in \mathbb{N}$. Let S be a set of integers that has size m.
Then, there exists a unique m-tuple $(s_1, s_2, \ldots, s_m)$ of integers satisfying

$$\{s_1, s_2, \ldots, s_m\} = S \qquad \text{and} \qquad s_1 < s_2 < \cdots < s_m.$$

This is just saying that the elements of a finite set of integers can always be listed in (strictly) increasing order, and this can be done in exactly one way. A rigorous proof of this can be found in various places (see, e.g., the 2019 notes for a reference), but intuitively this fact is self-evident.

Proposition 2.9.6 has an analogue for multisets:[1]

Proposition 2.9.6. Let T be a set of integers. Let $m \in \mathbb{N}$. Let S be a multisubset[2] of T that has size m.
Then, there exists a unique m-tuple $(s_1, s_2, \ldots, s_m)$ of integers in T satisfying

$$\{s_1, s_2, \ldots, s_m\}_{\text{multi}} = S \qquad \text{and} \qquad s_1 \leq s_2 \leq \cdots \leq s_m.$$

If $(s_1, s_2, \ldots, s_m)$ is this m-tuple, then we shall write $S = \{s_1 \leq s_2 \leq \cdots \leq s_m\}_{\text{multi}}$.
Again, we shall not give a rigorous proof of this proposition (for such a proof, see the solution to Exercise 2.11.2 in the 2019 notes), as we believe it to be intuitively obvious.

Next, we define the union of two disjoint multisets (more precisely, of two multisubsets of two disjoint sets):

Definition 2.9.7. Let A and B be two disjoint sets.
Let X be a multisubset of A. Let Y be a multisubset of B.
Then, their **union** $X \cup Y$ is defined to be the multisubset of $A \cup B$ defined as follows:

[1] The analogy is slightly broken by the fact that we have not defined multisets per-se, but only defined multisubsets, so we now need to introduce a set T. But this difference is insubstantial.

[2] Of course, "under the hood", a multisubset of T is a function (from T to $\mathbb{N}$), so it would make more sense to use a lowercase letter such as f for it. But we regard multisubsets as analogues of subsets, and so we find it more natural to denote them by uppercase letters (such as S here).

If $X = \{x_1, x_2, \ldots, x_k\}_{\text{multi}}$ and $Y = \{y_1, y_2, \ldots, y_\ell\}_{\text{multi}}$, then

$$X \cup Y := \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_\ell\}_{\text{multi}} .$$

In other words, $X \cup Y$ (regarded as a function from $A \cup B$ to $\mathbb{N}$) is defined by

$$(X \cup Y)(u) = \begin{cases} X(u), & \text{if } u \in A; \\ Y(u), & \text{if } u \in B \end{cases} \qquad \text{for each } u \in A \cup B$$

(to make sense of this, recall that a multisubset is actually defined as a function to $\mathbb{N}$, so it can applied to an element).

For instance, let us take the union of the multisubset $\{1,1,2\}_{\text{multi}}$ of $\{1,2,3\}$ with the multisubset $\{4,5,5\}_{\text{multi}}$ of $\{4,5,6,7\}$:

$$\{1,1,2\}_{\text{multi}} \cup \{4,5,5\}_{\text{multi}} = \{1,1,2,4,5,5\}_{\text{multi}} .$$

Note that in Definition 2.9.7, we always have $|X \cup Y| = |X| + |Y|$ (where we let $|S|$ denote the size of a multisubset S). This is similar to the sum rule for disjoint sets.

We also observe a simple property of unions:

Lemma 2.9.8. Let A and B be two disjoint sets. Let Z be a multisubset of $A \cup B$. Then, we can uniquely write Z as $Z = X \cup Y$ for some multisubset X of A and some multisubset Y of B.

In other words, if A and B are two disjoint sets, then a multisubset of $A \cup B$ can be uniquely decomposed into a union of a multisubset of A with a multisubset of B. This is intuitively clear and also easy to prove (exercise for the reader).

Now, the promised application of multisets to sets:

Proposition 2.9.9 (Musiker and Propp 2007)**.** Let $m \in \mathbb{N}$ and $a, b \in \{0, 1, \ldots, m\}$. Then,

$$\begin{aligned} &(\# \text{ of lacunar subsets of } [2m] \text{ with exactly } a \text{ even and } b \text{ odd elements}) \\ &= \binom{m-a}{b} \cdot \binom{m-b}{a}. \end{aligned}$$

Proof of Proposition 2.9.9. Let $g := m - a - b + 1$.

The set $[2g]$ is the union of its two disjoint g-element subsets[3]

$$E := \{\text{even elements of } [2g]\} = \{2, 4, 6, \ldots, 2g\} \qquad \text{and}$$
$$O := \{\text{odd elements of } [2g]\} = \{1, 3, 5, \ldots, 2g-1\}.$$

Let S be a lacunar subset of $[2m]$ with exactly a even and b odd elements. Then, S can be written uniquely in the form $S = \{s_1 < s_2 < \cdots < s_{a+b}\}$ (since $|S| = a + b$). Since S is lacunar, any two consecutive elements s_i and s_{i+1} of S satisfy $s_i \leq s_{i+1} - 2$. Therefore,

$$s_1 - 0 \leq s_2 - 2 \leq s_3 - 4 \leq \cdots \leq s_{a+b} - 2(a+b-1).$$

Hence, we can define the multisubset

$$M_S := \{s_1 - 0 \leq s_2 - 2 \leq s_3 - 4 \leq \cdots \leq s_{a+b} - 2(a+b-1)\}_{\text{multi}}$$

of $[2g]$. (Why is this a multisubset of $[2g]$? Because its largest element is $\underbrace{s_{a+b}}_{\leq 2m} - 2(a+b-1) \leq 2m - 2(a+b-1) = 2g$.)

This multisubset M_S is obtained from S by lowering the smallest element by 0, the second-smallest by 2, the third-smallest by 4, and so on; note that the parities of all elements are clearly preserved under this operation. Hence, M_S has exactly a even and b odd elements (counted with multiplicities), since S has exactly a even and b odd elements. In other words, M_S has a elements from E and b elements from O.

Now, the sets E and O are disjoint, and our M_S is a multisubset of $[2g] = E \cup O$. Hence, by Lemma 2.9.8, we can uniquely write M_S as $M_S = E_S \cup O_S$, where E_S is a multisubset of E and where O_S is a multisubset of O. The result of the previous paragraph shows that the multisubset E_S has size a, and the multisubset O_S has size b.

Forget that we fixed S. Thus, for each lacunar subset S of $[2m]$ with exactly a even and b odd elements, we have constructed a size-a multisubset E_S of E and a size-b multisubset O_S of O. Hence, we can define a map

$$\text{from } \{\text{lacunar subsets of } [2m] \text{ with exactly } a \text{ even elements and } b \text{ odd elements}\}$$
$$\text{to } \{\text{size-}a \text{ multisubsets of } E\} \times \{\text{size-}b \text{ multisubsets of } O\},$$

which sends each S to the corresponding pair (E_S, O_S).

[3]There is a little gap here: The subsets E and O are not g-element sets if $g < 0$, because they are 0-element sets in this case!

Fortunately, this is no big hindrance. Proposition 2.9.9 is easy to prove in the case when $g < 0$ (just argue that both sides are 0). Thus, we can WLOG assume that $g \geq 0$, and then the argument works fine.

A moment of thought reveals that this map has an inverse[4], and thus is a bijection. Hence, the bijection principle shows that

$$\begin{aligned}
&\left|\{\text{lacunar subsets of } [2m] \text{ with exactly } a \text{ even elements and } b \text{ odd elements}\}\right| \\
&= \left|\{\text{size-}a \text{ multisubsets of } E\} \times \{\text{size-}b \text{ multisubsets of } O\}\right| \\
&= \underbrace{\left|\{\text{size-}a \text{ multisubsets of } E\}\right|}_{\substack{=\binom{a+g-1}{a} \\ \text{(by Corollary 2.9.3,} \\ \text{since } E \text{ is a } g\text{-element set)}}} \cdot \underbrace{\left|\{\text{size-}b \text{ multisubsets of } O\}\right|}_{\substack{=\binom{b+g-1}{b} \\ \text{(by Corollary 2.9.3,} \\ \text{since } O \text{ is a } g\text{-element set)}}} \\
&= \binom{a+g-1}{a} \cdot \binom{b+g-1}{b} \\
&= \binom{m-b}{a} \cdot \binom{m-a}{b} \qquad \left(\begin{array}{c} \text{since } g = m-a-b+1 \\ \text{and therefore } a+g-1 = m-b \\ \text{and } b+g-1 = m-a \end{array}\right) \\
&= \binom{m-a}{b} \cdot \binom{m-b}{a}.
\end{aligned}$$

Proposition 2.9.9 thus follows. □

Corollary 2.9.10. For any $m \in \mathbb{N}$, the Fibonacci number f_{2m+2} satisfies

$$f_{2m+2} = \sum_{a=0}^{m} \sum_{b=0}^{m} \binom{m-a}{b} \cdot \binom{m-b}{a}.$$

Proof. Let $m \in \mathbb{N}$. Then, Proposition 1.4.7 from Lecture 9 (applied to $n = 2m$)

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[4]*Proof.* Let us review what this map does to a given lacunar subset S:

- First, each element of S is lowered by an appropriate even number (the smallest element by 0, the second-smallest by 2, the third-smallest by 4, and so on).
- Then, the resulting $a + b$ numbers are collected in a multisubset of $[2g]$.
- Finally, this multisubset $[2g]$ is decomposed into a multisubset E_S of E and a multisubset O_S of O.

The inverse of this map just undoes these three steps:

- First, we combine our multisubsets E_S and O_S of E and O to form the union $E_S \cup O_S$.
- Then, we list the $a + b$ elements of this union in weakly increasing order (making sure to account for their multiplicities).
- Finally, we increase each of these $a + b$ elements by an appropriate even number (the smallest element by 0, the second-smallest by 2, the third-smallest by 4, and so on).

yields

$$
\begin{aligned}
& f_{2m+2} \\
& = (\text{\# of lacunar subsets of } [2m]) \\
& = \sum_{a=0}^{m} \sum_{b=0}^{m} \underbrace{(\text{\# of lacunar subsets of } [2m] \text{ with exactly } a \text{ even and } b \text{ odd elements})}_{\substack{= \binom{m-a}{b} \cdot \binom{m-b}{a} \\ \text{(by Proposition 2.9.9)}}} \\
& \qquad \left(\begin{array}{c} \text{by the sum rule, since the \# of even elements of a} \\ \text{subset of } [2m] \text{ is always an integer between 0 and } m, \\ \text{and so is its \# of odd elements} \end{array} \right) \\
& = \sum_{a=0}^{m} \sum_{b=0}^{m} \binom{m-a}{b} \cdot \binom{m-b}{a}.
\end{aligned}
$$

□

2.10. Multinomial coefficients

Now, we shall define the **multinomial coefficients**: a generalization of the binomial coefficients, or at least of those binomial coefficients $\binom{n}{k}$ that have $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$.

2.10.1. Definition and formulas

Definition 2.10.1. Let $n \in \mathbb{N}$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ be such that $n_1 + n_2 + \cdots + n_k = n$. Then, we define

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{n_1! n_2! \cdots n_k!}.$$

This number is called a **multinomial coefficient**. It is a rational number, but we will soon see that it is an integer.

Example 2.10.2. We have $2 + 3 + 2 = 7$ and thus

$$\binom{7}{2, 3, 2} = \frac{7!}{2! \cdot 3! \cdot 2!} = 210.$$

Note that we are defining the multinomial coefficient $\binom{n}{n_1, n_2, \ldots, n_k}$ only under the conditions stated in Definition 2.10.1. In particular, we are not defining it for negative or non-integer n; nor are we defining it when $n_1 + n_2 + \cdots + n_k \neq n$. As a consequence, the notation $\binom{n}{n_1, n_2, \ldots, n_k}$ for multinomial coefficients does not clash with the notation $\binom{n}{j}$ for binomial coefficients. (More precisely, the two notations do clash when $k = 1$, leaving the expression $\binom{n}{n}$ ambiguous; but it is easy to see that both possible meanings of $\binom{n}{n}$ equal 1, and thus the clash is harmless.)

Before we prove anything interesting about multinomial coefficients, we observe that they generalize the entries of Pascal's triangle:

Proposition 2.10.3. Let $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. Then, the binomial coefficient $\binom{n}{k}$ equals the multinomial coefficient $\binom{n}{k, n-k}$.

Proof. The factorial formula for BCs yields $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$, but the multinomial coefficient $\binom{n}{k, n-k}$ is defined to be $\dfrac{n!}{k!\,(n-k)!}$ as well. $\square$

Here are a few basic properties of multinomial coefficients:

Proposition 2.10.4. Let $n \in \mathbb{N}$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ be such that $n_1 + n_2 + \cdots + n_k = n$. Then:

(a) We have

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots, n_k} &= \prod_{i=1}^{k} \binom{n - n_1 - n_2 - \cdots - n_{i-1}}{n_i} \\ &= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3} \cdots \underbrace{\binom{n - n_1 - n_2 - \cdots - n_{k-1}}{n_k}}_{=1} \\ &= \prod_{i=1}^{k-1} \binom{n - n_1 - n_2 - \cdots - n_{i-1}}{n_i}.\end{aligned}$$

(b) We have $\binom{n}{n_1, n_2, \ldots, n_k} \in \mathbb{N}$.

(c) The multinomial coefficient $\binom{n}{n_1, n_2, \ldots, n_k}$ does not change when we permute $n_1, n_2, \ldots, n_k$.

(d) If $n > 0$, then

$$\binom{n}{n_1, n_2, \ldots, n_k} = \sum_{i=1}^{k} \underbrace{\binom{n}{n_1, n_2, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k}}_{\text{This is understood to be 0 if } n_i = 0}.$$

(We call this the **Multi-Pascal's recurrence**, since it generalizes Pascal's recurrence at least in the case of nonnegative integers.)

Proof. Easy exercises. (All the proofs appear in the 2019 notes: The proofs of **(a)** and **(b)** are Exercise 2.12.1 in the 2019 notes, whereas part **(c)** is Proposition 2.12.14 in the 2019 notes, and part **(d)** is Exercise 2.12.5 in the 2019 notes.) □

Next time, we will learn how the multinomial coefficients got their name, and what they count.

References

[BecSan18] Matthias Beck, Raman Sanyal, *Combinatorial Reciprocity Theorems: An Invitation To Enumerative Geometric Combinatorics*, Graduate Studies in Mathematics **195**, AMS 2018.
`https://matthbeck.github.io/crt.html`