Math 222 Fall 2022, Lecture 21: Binomial coefficients

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

2. Binomial coefficients (cont'd)

2.8. Compositions

Next, we shall count various types of tuples of positive integers. BCs (= binomial coefficients) will appear as answers rather often.

2.8.1. Compositions

How many ways are there to write the number 5 as a sum of 3 positive integers, if the order matters? There are 6 of them:

$$\begin{aligned} 5 &= 2+2+1 = 2+1+2 = 1+2+2 \\ &= 1+1+3 = 1+3+1 = 3+1+1. \end{aligned}$$

What if we replace 5 and 3 by two arbitrary nonnegative integers n and k ? So we want to count the k-tuples $(x_1, x_2, \ldots, x_k)$ of positive integers whose sum is $x_1 + x_2 + \cdots + x_k = n$. The following theorem answers this question:

Theorem 2.8.1. Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ be the set of all positive integers.
Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,

$$\begin{aligned} &\left(\# \text{ of } (x_1, x_2, \ldots, x_k) \in \mathbb{P}^k \text{ satisfying } x_1 + x_2 + \cdots + x_k = n\right) \\ &= \binom{n-1}{n-k} \qquad (1) \\ &= \begin{cases} \dbinom{n-1}{k-1}, & \text{if } n > 0; \\ [k=0], & \text{if } n = 0. \end{cases} \qquad (2) \end{aligned}$$

Before we prove this, let us introduce some terminology for these tuples that we are counting:

Definition 2.8.2. (a) A **composition** shall mean a tuple (i.e., finite list) of positive integers. (This has nothing to do with the composition $f \circ g$ of two maps f and g.)

(b) If $k \in \mathbb{N}$, then a **composition into** k **parts** shall mean a k-tuple of positive integers.

(c) If $n \in \mathbb{N}$, then a **composition of** n shall mean a tuple of positive integers whose sum is n.

(d) If $n, k \in \mathbb{N}$, then a **composition of** n **into** k **parts** shall mean a k-tuple of positive integers whose sum is n.

Example 2.8.3. **(a)** The compositions of 5 into 3 parts are

$$\begin{array}{ccc} (2,2,1), & (2,1,2), & (1,2,2), \\ (1,1,3), & (1,3,1), & (3,1,1) \end{array}$$

(as we saw above).

(b) The compositions of 3 are

$$(1,1,1), \qquad (1,2), \qquad (2,1), \qquad (3).$$

These are compositions into 3, 2, 2 and 1 parts, respectively.

(c) The only composition of 0 is the 0-tuple $()$; it is a composition into 0 parts.

Theorem 2.8.1 can now be restated as follows: For any $n, k \in \mathbb{N}$, we have

$$\begin{aligned} &(\text{\# of compositions of } n \text{ into } k \text{ parts}) \\ &= \binom{n-1}{n-k} \\ &= \begin{cases} \dbinom{n-1}{k-1}, & \text{if } n > 0; \\ [k=0], & \text{if } n = 0. \end{cases} \end{aligned}$$

Proof of Theorem 2.8.1. (See Theorem 2.10.1 in the 2019 notes for details.)

First, we WLOG assume that $n > 0$, since the $n = 0$ case is pretty trivial. Thus, $n \geq 1$, so that $n - 1 \in \mathbb{N}$.

We shall now prove that

$$(\text{\# of compositions of } n \text{ into } k \text{ parts}) = \binom{n-1}{k-1}.$$

To do so, we construct a bijection

$$C : \{\text{compositions of } n \text{ into } k \text{ parts}\} \to \{(k-1)\text{-element subsets of } [n-1]\}.$$

We define this bijection as follows: For any composition $(a_1, a_2, \ldots, a_k)$ of n into k parts, we set

$$\begin{aligned} C(a_1, a_2, \ldots, a_k) &:= \{a_1, \ \ a_1 + a_2, \ \ a_1 + a_2 + a_3, \ \ \ldots, \ \ a_1 + a_2 + \cdots + a_{k-1}\} \\ &= \{a_1 + a_2 + \cdots + a_i \ \mid \ i \in [k-1]\}. \end{aligned}$$

[1] This set $C(a_1, a_2, \ldots, a_k)$ is called the **partial sum set** of $(a_1, a_2, \ldots, a_k)$.

It is not hard to see that the map C is well-defined (indeed, if $(a_1, a_2, \ldots, a_k)$ is a composition of n into k parts, then the positivity of its entries $a_1, a_2, \ldots, a_k$ yields

$$\begin{aligned} 0 &< a_1 < a_1 + a_2 < a_1 + a_2 + a_3 < \cdots < a_1 + a_2 + \cdots + a_{k-1} \\ &< a_1 + a_2 + \cdots + a_k = n, \end{aligned}$$

which shows that $C(a_1, a_2, \ldots, a_k)$ is a $(k-1)$-element subset of $[n-1]$). Now, how do we see that this map C is really a bijection? The easiest way is to construct an inverse map. Explicitly, this inverse map C^{-1} sends every $(k-1)$-element subset $\{i_1 < i_2 < \cdots < i_{k-1}\}$ of $[n-1]$ to the composition

$$(i_1 - i_0,\ i_2 - i_1,\ i_3 - i_2,\ \ldots,\ i_{k-1} - i_{k-2},\ i_k - i_{k-1}),$$

where we set $i_0 := 0$ and $i_k := n$.

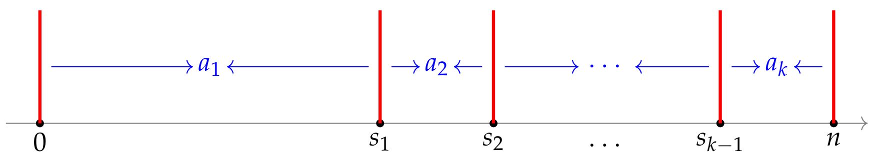
For a formal proof that all of this works, see §2.10.1 in the 2019 notes (which in turn refer to solved homework).

Anyway, we now know that C is a bijection. Thus, the bijection principle yields that

$$|\{\text{compositions of } n \text{ into } k \text{ parts}\}| = |\{(k-1)\text{-element subsets of } [n-1]\}|.$$

[1]Here is a geometric way to think of the map C:

Given a composition $(a_1, a_2, \ldots, a_k)$ of n, we can subdivide the interval $[0, n]$ on the real line into k blocks (i.e., subintervals) of lengths $a_1, a_2, \ldots, a_k$ (from left to right), as follows:



The set $C(a_1, a_2, \ldots, a_k)$ then consists of the $k-1$ points $s_1, s_2, \ldots, s_{k-1}$ at which these blocks begin and end (not counting the endpoints 0 and n of the big interval). Explicitly, it is easy to see that $s_i = a_1 + a_2 + \cdots + a_i$ for each $i \in [k-1]$, so that we obtain the formula

$$C(a_1, a_2, \ldots, a_k) = \{a_1 + a_2 + \cdots + a_i \mid i \in [k-1]\}$$

by which we have defined C. Most properties of the map C become clear by looking at this geometric interpretation.

In other words,

$$\begin{aligned}
&(\text{\# of compositions of } n \text{ into } k \text{ parts}) \\
&= (\text{\# of } (k-1)\text{-element subsets of } [n-1]) \\
&= \binom{n-1}{k-1} \qquad \left(\begin{array}{c}\text{by the combinatorial interpretation of BCs,} \\ \text{since } n-1 \in \mathbb{N}\end{array}\right) \\
&= \binom{n-1}{(n-1)-(k-1)} \qquad (\text{by the symmetry of BCs, since } n-1 \in \mathbb{N}) \\
&= \binom{n-1}{n-k}.
\end{aligned}$$

Thus, we have proved Theorem 2.8.1. □

2.8.2. Binary compositions

Here is a variant of Theorem 2.8.1 in which the entries of the tuples are supposed to be 0's and 1's instead of being positive integers:

Theorem 2.8.4. Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then,

$$\begin{aligned}
&\left(\text{\# of } (x_1, x_2, \ldots, x_k) \in \{0,1\}^k \text{ satisfying } x_1 + x_2 + \cdots + x_k = n\right) \\
&= \binom{k}{n}.
\end{aligned}$$

Proof. (See Theorem 2.10.4 in the 2019 notes for details.)

A k-tuple $(x_1, x_2, \ldots, x_k) \in \{0,1\}^k$ satisfies $x_1 + x_2 + \cdots + x_k = n$ if and only if it consists of n many 1's and $k-n$ many 0's (because $x_1 + x_2 + \cdots + x_k$ is just the # of 1's in $(x_1, x_2, \ldots, x_k)$). Thus, we can construct such a k-tuple by choosing the positions in which its n many 1's will be placed. This amounts to choosing an n-element subset of $[k]$, and this can be done in $\dbinom{k}{n}$ many ways. □

2.8.3. Weak compositions

One particularly useful variant of compositions are the so-called **weak compositions**. These are defined as tuples of nonnegative integers (i.e., they differ from compositions in that 0 is allowed as an entry). For example, the weak compositions of 2 into 3 parts are

$$\begin{array}{ccc}
(1,1,0), & (1,0,1), & (0,1,1), \\
(0,0,2), & (0,2,0), & (2,0,0).
\end{array}$$

Let us count weak compositions of n into k parts:

Theorem 2.8.5. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,

$$\begin{aligned}
&\left(\text{\# of } (x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \text{ satisfying } x_1 + x_2 + \cdots + x_k = n\right) \\
&= \binom{n+k-1}{n} \\
&= \begin{cases} \dbinom{n+k-1}{k-1}, & \text{if } k > 0; \\ [n=0], & \text{if } k = 0. \end{cases}
\end{aligned} \tag{3}$$

Proof of Theorem 2.8.5. Let $\mathbb{P} = \{1, 2, 3, \ldots\}$. There is a bijection

$$\begin{aligned}
&\text{from } \left\{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \mid x_1 + x_2 + \cdots + x_k = n\right\} \\
&\text{to } \left\{(x_1, x_2, \ldots, x_k) \in \mathbb{P}^k \mid x_1 + x_2 + \cdots + x_k = n + k\right\}
\end{aligned}$$

which sends

$$\begin{aligned}
&\text{each weak composition } (x_1, x_2, \ldots, x_k) \\
&\text{to the composition } (x_1 + 1, x_2 + 1, \ldots, x_k + 1)
\end{aligned}$$

(that is, which adds 1 to each entry). Thus, by the bijection principle, we have

$$\begin{aligned}
&\left(\text{\# of } (x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \text{ satisfying } x_1 + x_2 + \cdots + x_k = n\right) \\
&= \left(\text{\# of } (x_1, x_2, \ldots, x_k) \in \mathbb{P}^k \text{ satisfying } x_1 + x_2 + \cdots + x_k = n + k\right) \\
&= \binom{n+k-1}{n+k-k} \qquad \text{(by (1), applied to } n+k \text{ instead of } k) \\
&= \binom{n+k-1}{n}.
\end{aligned}$$

Using the symmetry of BCs (and a bit of case analysis to deal with the case of $k = 0$), we can furthermore rewrite this as

$$\begin{cases} \dbinom{n+k-1}{k-1}, & \text{if } k > 0; \\ [n=0], & \text{if } k = 0. \end{cases}$$

Thus, Theorem 2.8.5 is proved. □

Remark 2.8.6. Note that weak compositions appear in abstract algebra.

Indeed, consider the polynomial ring $\mathbb{R}[x_1, x_2, \ldots, x_k]$ in k indeterminates. The degree-n part of this polynomial ring (i.e., the vector space of polynomials that are homogeneous of degree n) is spanned by all monomials of degree

n in the indeterminates $x_1, x_2, \ldots, x_k$. These monomials have the form

$$x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}, \qquad \text{where } a_1, a_2, \ldots, a_k \in \mathbb{N} \text{ and } a_1 + a_2 + \cdots + a_k = n.$$

So these monomials are in one-to-one correspondence with the weak compositions of n into k parts. Thus, the dimension of the degree-n part of the polynomial ring is the # of these weak compositions. According to (3), this # is $\dbinom{n+k-1}{n}$.

2.8.4. Some other composition counting problems

Here is an assortment of other results about counting compositions. (See §2.10.4 in the 2019 notes for proofs.)

First, let us count **all** compositions of a number n (as opposed to just those having a given length k):

Proposition 2.8.7. Let $n \in \mathbb{N}$. Then,

$$(\text{\# of compositions of } n) = \begin{cases} 2^{n-1}, & \text{if } n > 0; \\ 1, & \text{if } n = 0. \end{cases}$$

Now, let us count compositions whose entries are 1's and 2's:

Proposition 2.8.8. Let $n \in \mathbb{N}$. A $\{1,2\}$**-composition of** n shall mean a composition $(x_1, x_2, \ldots, x_k)$ of n such that $x_1, x_2, \ldots, x_k \in \{1,2\}$. Then:

(a) The # of $\{1,2\}$-compositions of n is the Fibonacci number f_{n+1}.

(b) For any $k \in \mathbb{N}$, the # of $\{1,2\}$-compositions of n into k parts is $\dbinom{k}{n-k}$.

Next, we can try to count weak compositions of n into k parts whose entries all belong to $\{0, 1, \ldots, p-1\}$ for a given number p. When $p = 2$, this has already been done (see Theorem 2.8.4 above), and the answer was $\dbinom{k}{n}$. Unfortunately, no such simple answer exists for general p, but at least we can write it as a finite sum:

Proposition 2.8.9. Let $p \in \mathbb{N}$. Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then,

$$\left(\text{\# of } (x_1, x_2, \ldots, x_k) \in \{0, 1, \ldots, p-1\}^k \text{ such that } x_1 + x_2 + \cdots + x_k = n\right)$$
$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n - pj + k - 1}{n - pj}.$$

Finally, instead of counting compositions of n, let us sum their lengths:

Proposition 2.8.10. Let n be a positive integer. Then, the sum of the lengths of all compositions of n is $(n+1)\,2^{n-2}$.

See the 2019 notes for proofs of all of these propositions (they make nice exercises).

A much harder problem is counting **partitions**: compositions whose entries are arranged in weakly decreasing order (i.e., from highest to lowest). For instance, the partitions of 5 are

$$(5)\,, \qquad (4,1)\,, \qquad (3,2)\,, \qquad (3,1,1)\,, \qquad (2,2,1)\,,$$
$$(2,1,1,1)\,, \qquad (1,1,1,1,1)\,.$$

There are 7 of them, which is not a very simple number! In general, there is no explicit formula for the # of partitions of n, but there is at least one beautiful recurrence and a whole lot of interesting identities. Partition identities are a subject of their own. We will learn more about partitions in §3.7 (Lecture 25).

2.9. Multisubsets

Another counting problem that naturally leads to BCs is to count **multisubsets** of a given set having a given size. Let us introduce the notion of a multisubset.

2.9.1. Definitions

Sets in mathematics are rather binary objects: A set either contains an element or does not. It cannot "contain this element twice". This is useful in some situations and limiting in others. For the latter situations, the notion of a **multiset** has been invented. Essentially, a multiset is "a set that can contain some elements multiple times".

Instead of defining multisets per se, we will here only define the notion of a **multisubset**, which is essentially a multiset whose elements come from a given set T. Our multisubsets shall always be finite, so they can only contain finitely many elements, and each element is contained only finitely many times. Formally speaking, a multisubset of a set T is determined by specifying how often it contains each given element of T. In other words, it is determined by the function $f : T \to \mathbb{N}$ which sends each $t \in T$ to the multiplicity of t in this multisubset (i.e., the number of times that t is contained in the multisubset).

For a rigorous definition, we simply define a multisubset to be such a function:

Definition 2.9.1. Let T be a set.

A **multisubset** of T is formally defined as a map $f : T \to \mathbb{N}$ such that only finitely many $t \in T$ satisfy $f(t) \neq 0$. [2]

Informally, we regard such a map $f : T \to \mathbb{N}$ as a way to encode a "set with multiplicities" – namely, the "set" in which each $t \in T$ appears $f(t)$ many times. Accordingly, we will use the notation $\{a_1, a_2, \ldots, a_k\}_{\text{multi}}$ for this multisubset f, where $(a_1, a_2, \ldots, a_k)$ is a list of elements of T such that each $t \in T$ appears in this list exactly $f(t)$ many times.

For example, if $T = [8]$, then the map from $[8]$ to $\mathbb{N}$ that is given in two-line notation as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 0 & 3 & 2 & 1 & 0 & 1 \end{pmatrix}$$

is a multisubset of $[8]$, and it can be written as

$$\{1, 4, 4, 4, 5, 5, 6, 8\}_{\text{multi}}$$

(or as $\{4, 5, 4, 4, 5, 6, 1, 8\}_{\text{multi}}$, or in many other ways). It contains the element 1 once, the element 4 thrice, the element 5 twice, the element 6 once and the element 8 once.

Note that $\{1, 1\}_{\text{multi}} \neq \{1\}_{\text{multi}}$, even though $\{1, 1\} = \{1\}$. Thus, multisubsets are basically "subsets with multiplicities". However, $\{1, 2\}_{\text{multi}} = \{2, 1\}_{\text{multi}}$, since multisets don't come with an ordering of their elements.

The **size** of a multisubset is easily defined:

Definition 2.9.2. Let T be a set. Let f be a multisubset of T. Then, the **size** of f is defined to be the number $\sum\limits_{t \in T} f(t) \in \mathbb{N}$.

This means that the size of a multisubset f is the number of elements of f, counted with multiplicities (i.e., the sum of the multiplicities of all elements of f). Equivalently, any multisubset that has the form $\{a_1, a_2, \ldots, a_k\}_{\text{multi}}$ has size k (no matter whether the elements $a_1, a_2, \ldots, a_k$ are distinct or not).

Next time, we will answer the following question: How many **multi**subsets of a given set T have a given size k ? (The analogous question about subsets has been answered back in Lecture 6.)

[2] The latter condition is meant to ensure that our multisubset is finite even if T itself might be infinite.