Math 222 Fall 2022, Lecture 19: Binomial coefficients

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

2. Binomial coefficients (cont'd)

2.7. The principle of inclusion and exclusion (aka Sylvester's sieve formula)

The principle of inclusion and exclusion is probably the last of the major counting principles. We will state it in four equivalent forms and prove it soon after. Then, in the next lecture, we will see some of its applications.

2.7.1. The principles

Playing around with finite sets, one may spot a few patterns. For instance, the size of the union of a few finite sets can be computed using the sizes of their intersections:

- For any two finite sets A and B, we have

$$|A \cup B| = |A| + |B| - |A \cap B| . \tag{1}$$

- For any three finite sets A, B and C, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| . \tag{2}$$

- For any four finite sets A, B, C and D, we have

$$\begin{aligned} & |A \cup B \cup C \cup D| \\ & = |A| + |B| + |C| + |D| \\ & \qquad - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & \qquad + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & \qquad - |A \cap B \cap C \cap D| . \end{aligned} \tag{3}$$

These three equalities all follow the same pattern: On the left is the size of a union of some finite sets; on the right are the sizes of their intersections (not just the intersection of all the sets, but also the intersections of some of them, such as $A \cap B \cap D$ or $B \cap C$ or even B). This pattern remains valid for any (finite) number of finite sets:

Theorem 2.7.1. Let $n \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_n$ be n finite sets. Then,

$$\begin{aligned} &|A_1 \cup A_2 \cup \cdots \cup A_n| \\ &= \sum_{m=1}^{n} (-1)^{m-1} \sum_{\substack{(i_1,i_2,\ldots,i_m)\in[n]^m; \\ i_1<i_2<\cdots<i_m}} \left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}\right|. \end{aligned} \tag{4}$$

The nested sum on the right hand side of (4) looks scary, so let us write it out: It is

$$\begin{aligned} &\underbrace{|A_1| + |A_2| + \cdots + |A_n|}_{\substack{\text{the sizes of all } A_i \\ \text{(summed with + signs)}}} \\ &\qquad \underbrace{- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n|}_{\substack{\text{the sizes of all intersections } A_i \cap A_j \text{ (with } i<j) \\ \text{(summed with } - \text{ signs)}}} \\ &\qquad \underbrace{+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|}_{\substack{\text{the sizes of all intersections } A_i \cap A_j \cap A_k \text{ (with } i<j<k) \\ \text{(summed with + signs)}}} \\ &\qquad \pm \cdots \\ &\qquad + \qquad \underbrace{(-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|}_{\substack{\text{the size of the intersection } A_1 \cap A_2 \cap \cdots \cap A_n \\ \text{(summed with a + or } - \text{ sign depending on the parity of } n)}} \qquad . \end{aligned}$$

Thus, it is a sum that contains the sizes of all possible intersections of some of the sets $A_1, A_2, \ldots, A_n$, each with a $+$ or $-$ sign depending on how many sets are being intersected. Clearly, the right hand sides of (1), (2) and (3) are the particular cases of this sum obtained for $n = 2$, $n = 3$ and $n = 4$, respectively (and with $A_1, A_2, A_3, \ldots$ renamed as $A, B, C, \ldots$).

Theorem 2.7.1 is known as the **Principle of Inclusion and Exclusion** (short: **PIE**) or as **Sylvester's sieve formula**. We will not prove it right away, but rather rewrite it in several other forms, some of which are easier to prove and more useful.

For our first restatement, we need a notation from set theory:

Definition 2.7.2. Let I be a nonempty set. For each $i \in I$, let A_i be any set. Then, we set

$$\bigcap_{i \in I} A_i := \{x \mid x \in A_i \text{ for each } i \in I\}.$$

This set $\bigcap\limits_{i \in I} A_i$ is called the **intersection** of the sets A_i for $i \in I$.

For example:

- If $I = \{i_1, i_2, \ldots, i_m\}$ is a finite set, then $\bigcap\limits_{i \in I} A_i = A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$.
- If I is a bunch of people, and if $A_i = \{\text{friends of } i\}$ for each $i \in I$, then $\bigcap\limits_{i \in I} A_i = \{x \mid x \text{ is friends with } \textbf{everyone} \text{ in } I\}$.

Note the similarity between this notation $\bigcap\limits_{i \in I} A_i$ and the well-known notations for finite sums ($\sum\limits_{i \in I} a_i$) and finite products ($\prod\limits_{i \in I} a_i$). There is also an analogous notation $\bigcup\limits_{i \in I} A_i$ for the **union** of some sets.

Now, we can rewrite the RHS of Theorem 2.7.1 as follows:

Proposition 2.7.3. Let $n \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_n$ be n finite sets. Then,

$$\sum_{m=1}^{n} (-1)^{m-1} \sum_{\substack{(i_1, i_2, \ldots, i_m) \in [n]^m; \\ i_1 < i_2 < \cdots < i_m}} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m} \right| = \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|.$$

Example 2.7.4. If $n = 3$, then the claim of Proposition 2.7.3 takes the form

$$\begin{aligned}
& (-1)^{1-1} \left(|A_1| + |A_2| + |A_3|\right) \\
& \qquad + (-1)^{2-1} \left(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\right) \\
& \qquad + (-1)^{3-1} |A_1 \cap A_2 \cap A_3| \\
& = (-1)^{|\{1\}|-1} \left| \bigcap_{i \in \{1\}} A_i \right| + (-1)^{|\{2\}|-1} \left| \bigcap_{i \in \{2\}} A_i \right| + (-1)^{|\{3\}|-1} \left| \bigcap_{i \in \{3\}} A_i \right| \\
& \qquad + (-1)^{|\{1,2\}|-1} \left| \bigcap_{i \in \{1,2\}} A_i \right| + (-1)^{|\{1,3\}|-1} \left| \bigcap_{i \in \{1,3\}} A_i \right| \\
& \qquad + (-1)^{|\{2,3\}|-1} \left| \bigcap_{i \in \{2,3\}} A_i \right| + (-1)^{|\{1,2,3\}|-1} \left| \bigcap_{i \in \{1,2,3\}} A_i \right|
\end{aligned}$$

(since the subsets I of $[n]$ that satisfy $I \neq \varnothing$ are $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$). You can easily confirm this by checking that the addends on the left hand side (after the parentheses are expanded) are precisely the addends on the right hand side.

Proof of Proposition 2.7.3. Essentially, the reason is the following: Both the LHS and the RHS are the sum of the sizes of all possible intersections of some of our n sets $A_1, A_2, \ldots, A_n$, with appropriate signs (namely, a "$+$" sign whenever we are intersecting an odd # of sets, and a "$-$" sign whenever we are intersecting an even # of sets). They differ only in their organization (the RHS is taking this sum directly, whereas the LHS splits it up according to the # of sets being intersected, and labels the possible intersections by their indices in increasing order).

For the details of this proof, see the 2019 notes (Proposition 2.9.4). □

We can now restate the Principle of Inclusion and Exclusion as follows:

Theorem 2.7.5 (Principle of Inclusion and Exclusion, union form)**.** Let $n \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_n$ be n finite sets. Then,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right| . \tag{5}$$

We will soon prove this theorem (and the previous one). But first, we shall restate it further, by focussing not on the union $A_1 \cup A_2 \cup \cdots \cup A_n$ of our n sets, but rather on the complement of this union (i.e., on everything that is **not** in this union). To make sense of this complement, we need to introduce a "universe" set U that contains all of our n sets $A_1, A_2, \ldots, A_n$ as subsets:

Theorem 2.7.6 (Principle of Inclusion and Exclusion, complement form)**.** Let $n \in \mathbb{N}$. Let U be a finite set. Let $A_1, A_2, \ldots, A_n$ be n subsets of U. Then,

$$|U \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)| = \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| . \tag{6}$$

Here, the "empty" intersection $\bigcap_{i \in \varnothing} A_i$ is understood to mean the set U.

Example 2.7.7. Let U be a finite set, and let A_1 and A_2 be two subsets of U.

Then, Theorem 2.7.6 (applied to $n = 2$) says that

$$\begin{aligned}
\left|U \setminus (A_1 \cup A_2)\right| &= \sum_{I \subseteq [2]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| \\
&= \underbrace{(-1)^{|\varnothing|}}_{=1} \left| \underbrace{\bigcap_{i \in \varnothing} A_i}_{\substack{=U \\ \text{(by definition)}}} \right| + \underbrace{(-1)^{|\{1\}|}}_{=-1} \left| \underbrace{\bigcap_{i \in \{1\}} A_i}_{=A_1} \right| \\
&\qquad + \underbrace{(-1)^{|\{2\}|}}_{=-1} \left| \underbrace{\bigcap_{i \in \{2\}} A_i}_{=A_2} \right| + \underbrace{(-1)^{|\{1,2\}|}}_{=1} \left| \underbrace{\bigcap_{i \in \{1,2\}} A_i}_{=A_1 \cap A_2} \right| \\
&\qquad \left(\text{since the subsets of } [2] \text{ are } \varnothing, \{1\}, \{2\}, \{1,2\}\right) \\
&= |U| - |A_1| - |A_2| + |A_1 \cap A_2| .
\end{aligned}$$

Note that $U \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)$ is the set of all elements of U that belong to **none** of $A_1, A_2, \ldots, A_n$.

Finally, let us restate Theorem 2.7.6 in yet another form (which is both easiest to prove and most convenient for applications):

Theorem 2.7.8. Let $n \in \mathbb{N}$. Let U be a finite set. Let $A_1, A_2, \ldots, A_n$ be n subsets of U. Then,[1]

$$\begin{aligned}
&(\# \text{ of all } s \in U \text{ such that } s \notin A_i \text{ for all } i \in [n]) \\
&= \sum_{I \subseteq [n]} (-1)^{|I|} (\# \text{ of all } s \in U \text{ such that } s \in A_i \text{ for all } i \in I) .
\end{aligned}$$

2.7.2. The cancellation lemma

We have now stated the Principle of Inclusion and Exclusion in four different forms. Soon we shall prove them. The proof will rely on the following crucial

[1] Note that the condition "$s \in A_i$ for all $i \in I$" is vacuously true if the set I is empty. Thus, if I is empty, then

$$(\# \text{ of all } s \in U \text{ such that } s \in A_i \text{ for all } i \in I) = (\# \text{ of all } s \in U) = |U| .$$

lemma[2]:

Proposition 2.7.9 (simple cancellation lemma)**.** Let S be a finite set. Then,

$$\sum_{I \subseteq S} (-1)^{|I|} = [S = \varnothing] .$$

Example 2.7.10. The subsets of $\{1,2\}$ are $\varnothing$, $\{1\}$, $\{2\}$ and $\{1,2\}$. Thus, applying Proposition 2.7.9 to $S = \{1,2\}$, we find

$$\underbrace{(-1)^{|\varnothing|}}_{=1} + \underbrace{(-1)^{|\{1\}|}}_{=-1} + \underbrace{(-1)^{|\{2\}|}}_{=-1} + \underbrace{(-1)^{|\{1,2\}|}}_{=1} = [\{1,2\} = \varnothing] .$$

Indeed, both sides of this equality are 0 (the left hand side because the addends cancel; the right hand side because $\{1,2\} \neq \varnothing$).

Proof of Proposition 2.7.9. The case $S = \varnothing$ is clear (since $1 = 1$). So we WLOG assume that $S \neq \varnothing$. Therefore, there exists some $g \in S$. Pick such a g.

A subset I of S will be called **green** if it satisfies $g \in I$, and will be called **red** otherwise. Then, there is a bijection

$$\begin{aligned} \{\text{green subsets}\} &\to \{\text{red subsets}\} , \\ J &\mapsto J \setminus \{g\} , \end{aligned}$$

whose inverse map is

$$\begin{aligned} \{\text{red subsets}\} &\to \{\text{green subsets}\} , \\ J &\mapsto J \cup \{g\} . \end{aligned}$$

This bijection allows us to pair up each green subset of S with a red subset of S (namely, each green subset J gets paired up with the red subset $J \setminus \{g\}$). In each pair thus obtained, the sizes of the two subsets differ by 1, so the corresponding $(-1)^{|I|}$ terms add up to 0. Thus, we have paired up all the addends in the sum $\sum_{I \subseteq S} (-1)^{|I|}$ in such a way that each pair sums to 0 (since the two addends in the pair cancel each other). Therefore, the whole sum is 0. But $[S = \varnothing]$ is also 0, since $S \neq \varnothing$. So we obtain $\sum_{I \subseteq S} (-1)^{|I|} = [S = \varnothing]$, as desired.

(See the 2019 notes (Proposition 2.9.10) for a formalized version of this proof, as well as for another proof.) $\square$

The reason for the word "simple" in the name of Proposition 2.7.9 is the existence of the following, slightly more general, cancellation lemma:

[2]We again use the Iverson bracket notation.

Proposition 2.7.11 (cancellation lemma)**.** Let S be a finite set. Let T be a subset of S. Then,

$$\sum_{\substack{I \subseteq S;\\ T \subseteq I}} (-1)^{|I|} = (-1)^{|T|} [S = T].$$

Proof. See Exercise 2.9.1 in the 2019 notes. □

For instance, if $S = [4]$ and $T = [2]$, then Proposition 2.7.11 is saying that

$$(-1)^{|\{1,2\}|} + (-1)^{|\{1,2,3\}|} + (-1)^{|\{1,2,4\}|} + (-1)^{|\{1,2,3,4\}|} = 0.$$

2.7.3. The proofs

We are now ready to prove the Principle of Inclusion and Exclusion in all its forms. We begin with the last form:

Theorem 2.7.12 (Theorem 2.7.8, repeated for convenience)**.** Let $n \in \mathbb{N}$. Let U be a finite set. Let $A_1, A_2, \ldots, A_n$ be n subsets of U. Then,

$$\begin{aligned} &(\text{\# of all } s \in U \text{ such that } s \notin A_i \text{ for all } i \in [n]) \\ &= \sum_{I \subseteq [n]} (-1)^{|I|} (\text{\# of all } s \in U \text{ such that } s \in A_i \text{ for all } i \in I). \end{aligned}$$

Proof. This will involve some set juggling. To gain a bit of useful intuition, you may want to anthropomorphize some of the sets and elements involved. For instance, you can think

- of U as a **u**niversity, defined (somewhat reductionistically) as a set of **s**tudents $s \in U$;
- of $A_1, A_2, \ldots, A_n$ as **a**ssociations, each of which contains some students;
- of a subset $I \subseteq [n]$ as a choice of associations (none if I is empty, or all of them if $I = [n]$) to be **i**ntersected.

The claim of Theorem 2.7.8 then can be rewritten as follows:

$$\begin{aligned} &(\text{\# of all students } s \text{ that belong to no association}) \\ &= \sum_{I \subseteq [n]} (-1)^{|I|} (\text{\# of all } s \in U \text{ that belong to all associations } A_i \text{ with } i \in I). \end{aligned}$$

We shall prove this by the following strategy: We fix a single student $s \in U$, and we compare its contribution to the # on the LHS with its contribution to the sum on the RHS.

The rigorous way to do this is to recall the roll-call principle (Lecture 11, Proposition 1.6.3 **(b)**). This principle yields

$$(\# \text{ of all } s \in U \text{ such that } s \notin A_i \text{ for all } i \in [n]) = \sum_{s \in U} [s \notin A_i \text{ for all } i \in [n]] . \tag{7}$$

For the same reason, if $I \subseteq [n]$, then

$$(\# \text{ of all } s \in U \text{ such that } s \in A_i \text{ for all } i \in I) = \sum_{s \in U} [s \in A_i \text{ for all } i \in I] .$$

Thus,

$$\begin{aligned}
&\sum_{I \subseteq [n]} (-1)^{|I|} \underbrace{(\# \text{ of all } s \in U \text{ such that } s \in A_i \text{ for all } i \in I)}_{= \sum_{s \in U} [s \in A_i \text{ for all } i \in I]} \\
&= \sum_{I \subseteq [n]} (-1)^{|I|} \sum_{s \in U} [s \in A_i \text{ for all } i \in I] \\
&= \sum_{I \subseteq [n]} \sum_{s \in U} (-1)^{|I|} [s \in A_i \text{ for all } i \in I] \\
&= \sum_{s \in U} \sum_{I \subseteq [n]} (-1)^{|I|} [s \in A_i \text{ for all } i \in I] \qquad (8)
\end{aligned}$$

(here, we have interchanged the two summation signs). Let us now simplify the inner sum here.

We fix a student $s \in U$, and we let

$$P_s := \{ i \in [n] \mid s \in A_i \} .$$

We shall refer to P_s as the "**p**assport of s", as it records which of our n associations $A_1, A_2, \ldots, A_n$ the student s belongs to. Hence, a subset I of $[n]$ satisfies "$s \in A_i$ for all $i \in I$" if and only if it satisfies $I \subseteq P_s$. Therefore, for any subset I of $[n]$, we have

$$[s \in A_i \text{ for all } i \in I] = [I \subseteq P_s]$$

(since equivalent statements have the same truth value). Therefore,

$$\begin{aligned}
&\sum_{I\subseteq[n]} (-1)^{|I|} \underbrace{[s \in A_i \text{ for all } i \in I]}_{=[I\subseteq P_s]} \\
&= \sum_{I\subseteq[n]} (-1)^{|I|} [I \subseteq P_s] \\
&= \sum_{\substack{I\subseteq[n];\\ I\subseteq P_s}} (-1)^{|I|} \underbrace{[I \subseteq P_s]}_{=1} + \sum_{\substack{I\subseteq[n];\\ I\not\subseteq P_s}} (-1)^{|I|} \underbrace{[I \subseteq P_s]}_{=0} \\
&\qquad \left(\begin{array}{c}\text{here, we have split our sum into two parts:}\\ \text{one containing the addends for which } I \subseteq P_s,\\ \text{and one containing all the other addends}\end{array}\right) \\
&= \sum_{\substack{I\subseteq[n];\\ I\subseteq P_s}} (-1)^{|I|} + \underbrace{\sum_{\substack{I\subseteq[n];\\ I\not\subseteq P_s}} (-1)^{|I|} 0}_{=0} = \sum_{\substack{I\subseteq[n];\\ I\subseteq P_s}} (-1)^{|I|} \\
&= \sum_{I\subseteq P_s} (-1)^{|I|} \qquad \left(\begin{array}{c}\text{since the subsets } I \text{ of } [n] \text{ satisfying } I \subseteq P_s\\ \text{are precisely the subsets of } P_s \text{ (because } P_s \subseteq [n]\text{)}\end{array}\right) \\
&= [P_s = \varnothing] \qquad \text{(by Proposition 2.7.9, applied to } S = P_s) \\
&= [s \notin A_i \text{ for each } i \in [n]]
\end{aligned} \tag{9}$$

(because the passport P_s of s is empty if and only if s belongs to no association, i.e., if and only if we have $s \notin A_i$ for each $i \in [n]$).

Forget that we fixed s. We have thus proved (9) for each $s \in U$. Now, (8) becomes

$$\begin{aligned}
&\sum_{I\subseteq[n]} (-1)^{|I|} (\text{\# of all } s \in U \text{ such that } s \in A_i \text{ for all } i \in I) \\
&= \sum_{s\in U} \underbrace{\sum_{I\subseteq[n]} (-1)^{|I|} [s \in A_i \text{ for all } i \in I]}_{\substack{=[s\notin A_i \text{ for each } i\in[n]]\\ \text{(by (9))}}} \\
&= \sum_{s\in U} [s \notin A_i \text{ for each } i \in [n]] \\
&= (\text{\# of all } s \in U \text{ such that } s \notin A_i \text{ for each } i \in [n]) \qquad (\text{by (7)}).
\end{aligned}$$

This proves Theorem 2.7.8. □

As we said, the four versions of the Principle of Inclusion and Exclusion are restatements of one another, so that, having proved one of them, we can easily derive the other three:

Proof of Theorem 2.7.6. Theorem 2.7.6 is equivalent to Theorem 2.7.8, since we have

$$\begin{aligned}
&(\text{\# of all } s \in U \text{ such that } s \notin A_i \text{ for all } i \in [n]) \\
&= (\text{\# of all } s \in U \text{ such that } s \notin A_1 \cup A_2 \cup \cdots \cup A_n) \\
&= (\text{\# of all } s \in U \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)) \\
&= |U \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)|
\end{aligned}$$

and since each $I \subseteq [n]$ satisfies[3]

$$\begin{aligned}
&(\text{\# of all } s \in U \text{ such that } s \in A_i \text{ for all } i \in I) \\
&= \left(\text{\# of all } s \in U \text{ such that } s \in \bigcap_{i \in I} A_i\right) \\
&= \left|\bigcap_{i \in I} A_i\right| \qquad \left(\text{since } \bigcap_{i \in I} A_i \subseteq U\right).
\end{aligned}$$

□

Proof of Theorem 2.7.5. Let $U := A_1 \cup A_2 \cup \cdots \cup A_n$. Then, U is a finite set, and $A_1, A_2, \ldots, A_n$ are n subsets of U. Hence, Theorem 2.7.6 yields

$$|U \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)| = \sum_{I \subseteq [n]} (-1)^{|I|} \left|\bigcap_{i \in I} A_i\right|.$$

However, the left hand side of this equality is 0 (since $U \setminus (A_1 \cup A_2 \cup \cdots \cup A_n) = \varnothing$). Thus, this equality rewrites as

$$\begin{aligned}
0 &= \sum_{I \subseteq [n]} (-1)^{|I|} \left|\bigcap_{i \in I} A_i\right| = \underbrace{(-1)^{|\varnothing|}}_{=(-1)^0=1} \left|\underbrace{\bigcap_{i \in \varnothing} A_i}_{\substack{=U \\ \text{(by definition)}}}\right| + \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} \underbrace{(-1)^{|I|}}_{=-(-1)^{|I|-1}} \left|\bigcap_{i \in I} A_i\right| \\
&\qquad \text{(here, we have split off the addend for } I = \varnothing \text{ from the sum)} \\
&= |U| + \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} \left(-(-1)^{|I|-1}\right) \left|\bigcap_{i \in I} A_i\right| = |U| - \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} (-1)^{|I|-1} \left|\bigcap_{i \in I} A_i\right|.
\end{aligned}$$

In other words,

$$|U| = \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} (-1)^{|I|-1} \left|\bigcap_{i \in I} A_i\right|.$$

[3]To be fully precise, this argument works only for nonempty I. If I is empty, a separate (even more trivial) argument is needed here.

In view of $U = A_1 \cup A_2 \cup \cdots \cup A_n$, this is precisely the claim of Theorem 2.7.5. $\square$

Proof of Theorem 2.7.1. Proposition 2.7.3 shows that the right hand sides of (4) and (5) are identical. Hence, (4) follows from (5). Thus, Theorem 2.7.1 is proven. $\square$