Math 222 Fall 2022, Lecture 18: Binomial coefficients

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

2. Binomial coefficients (cont'd)

2.6. Combinatorial proofs

Combinatorial proofs are proofs using combinatorial reasoning (which, in this course here, usually means counting). The word is used mostly for contrast with algebraic proofs (which rely on manipulation of sums and products, or properties of polynomials, or similar algebraic tools). We have seen many combinatorial proofs in the previous lectures; today we will see some more.

Here are the two most frequent types of combinatorial proofs:

- A **proof by double counting** establishes an identity of the form $x = y$ by showing that x and y answer one and the same counting problem (i.e., by finding a set S such that $|S| = x$ and $|S| = y$). For instance, our proof of Theorem 1.4.12 in Lecture 10, as well as our partial proof of Proposition 2.2.1 in Lecture 14, are proofs by double counting.

- A **bijective proof** is a proof using the bijection principle[1]. The proof of Proposition 1.4.8 in Lecture 9 is an example of a bijective proof.

Today, we will see more examples of both types. (Actually, the boundary between the two types is rather blurry; often, a bijective proof can be translated into a proof by double counting or vice versa.) Note that this lecture is just a taste; many more combinatorial proofs can be found in [BenQui03] and [Loehr11].

2.6.1. The symmetry of Pascal's triangle

Recall the symmetry of the binomial coefficients (Lecture 6, Theorem 1.3.9):

Theorem 2.6.1. Let $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Then,

$$\binom{n}{k} = \binom{n}{n-k}.$$

We gave a simple proof of this using the factorial formula for BCs (= binomial coefficients), but we can give a combinatorial proof as well:

[1] Recall that the bijection principle says the following: If there is a bijection from some set X to some set Y, then $|X| = |Y|$.

Combinatorial proof of Theorem 2.6.1. By the combinatorial interpretation of BCs, we have

$$\binom{n}{k} = (\text{\# of } k\text{-element subsets of } [n]) . \qquad (1)$$

Similarly,

$$\binom{n}{n-k} = (\text{\# of } (n-k)\text{-element subsets of } [n]) . \qquad (2)$$

Now, the map

$$\{k\text{-element subsets of } [n]\} \to \{(n-k)\text{-element subsets of } [n]\} ,$$
$$S \mapsto [n] \setminus S$$

is well-defined (because if S is a k-element subset of $[n]$, then its complement $[n] \setminus S$ is an $(n-k)$-element subset of $[n]$) and bijective (indeed, it has an inverse map, which also sends each S to $[n] \setminus S$). Thus, it is a bijection. Hence, the bijection principle yields

$$|\{k\text{-element subsets of } [n]\}| = |\{(n-k)\text{-element subsets of } [n]\}| .$$

In other words,

$$(\text{\# of } k\text{-element subsets of } [n]) = (\text{\# of } (n-k)\text{-element subsets of } [n]) .$$

In view of (1) and (2), this rewrites as

$$\binom{n}{k} = \binom{n}{n-k}.$$

This proves Theorem 2.6.1 again. □

2.6.2. The Chu–Vandermonde identity

Recall the Chu–Vandermonde identity (Theorem 2.3.1 in Lecture 15), which says the following:

Theorem 2.6.2 (Chu–Vandermonde identity)**.** Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Then,

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}.$$

We proved this in Lecture 15. Let us now give a different proof:

Combinatorial proof of Theorem 2.6.2. First, we assume that $x, y \in \mathbb{N}$. (We will later handle the general case.)

Let Q be the set $\{1, 2, \ldots, x\} \cup \{-1, -2, \ldots, -y\}$. How many n-element subsets does Q have?[2]

We can answer this question in two ways (for a double-counting proof):

1st way: Clearly, Q has $\binom{x+y}{n}$ many n-element subsets (by the combinatorial interpretation of BCs, since Q is an $(x+y)$-element set).

2nd way: We can choose an n-element subset S of Q using the following procedure:

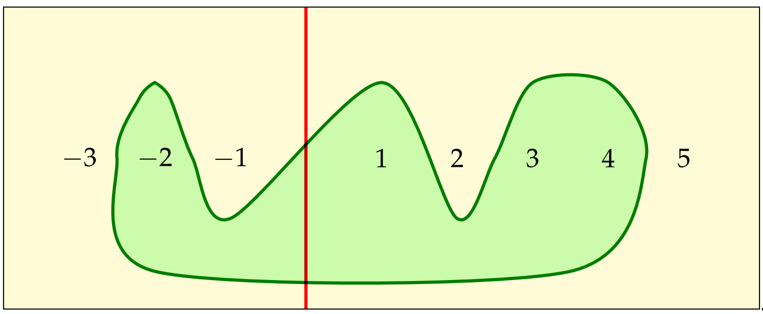
- First, we decide on the number of positive elements of S (that is, we decide how many elements of S will be positive). Let $k \in \{0, 1, \ldots, n\}$ be this number. Thus, the set S will have k positive elements and $n - k$ negative elements.
- Now, we choose the k positive elements of S. The # of ways to do this is $\binom{x}{k}$, because this amounts to choosing a k-element subset of the x-element set $\{1, 2, \ldots, x\}$.
- Finally, we choose the $n - k$ negative elements of S. The # of ways to do this is $\binom{y}{n-k}$, since this amounts to choosing an $(n-k)$-element subset of the y-element set $\{-1, -2, \ldots, -y\}$.

The total # of ways to perform this construction is

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}$$

(indeed, the summation sign $\sum\limits_{k=0}^{n}$ is owed to the choice of $k \in \{0, 1, \ldots, n\}$, and the addend $\binom{x}{k}\binom{y}{n-k}$ is the # of possibilities for S such that the # of positive elements of S is k).

[2]Here is an illustration of an n-element subset of Q (for $n = 4$ and $x = 5$ and $y = 3$):



The yellow rectangle is the set Q; the green blob is the subset. The vertical line separates the positive from the negative elements of Q (which will be crucial to the proof later on).

We have now found two answers to the question "how many n-element subsets does Q have?". The first answer is $\dbinom{x+y}{n}$; the second is $\sum\limits_{k=0}^{n} \dbinom{x}{k} \dbinom{y}{n-k}$. Since these two answers must be equal (because they answer the same question), we thus obtain

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}.$$

This proves Theorem 2.6.2 for all $x, y \in \mathbb{N}$. Now we can generalize it to $x, y \in \mathbb{R}$ as follows:

- Keep y fixed, and use the polynomial identity trick (Lecture 14, Corollary 2.2.5) to generalize from $x \in \mathbb{N}$ to $x \in \mathbb{R}$. (The polynomial identity trick must be applied to the polynomials $P = \dbinom{X+y}{n}$ and $Q = \sum\limits_{k=0}^{n} \dbinom{X}{k} \dbinom{y}{n-k}$ here.)
- Keep x fixed, and use the polynomial identity trick again to generalize from $y \in \mathbb{N}$ to $y \in \mathbb{R}$. (This time, the polynomial identity trick is applied to $P = \dbinom{x+X}{n}$ and $Q = \sum\limits_{k=0}^{n} \dbinom{x}{k} \dbinom{X}{n-k}$.)

(See §2.6.3 of the 2019 notes for the details of this argument.)

Thus, Theorem 2.6.2 is proved again. □

Remark 2.6.3. The proof we just gave was a proof by double counting. But we can easily transform it into a bijective proof.

Indeed, let us assume that $x, y \in \mathbb{N}$, and let Q be the set $\{1, 2, \ldots, x\} \cup \{-1, -2, \ldots, -y\}$.

For any set A and any $i \in \mathbb{N}$, let $\mathcal{P}_i(A)$ denote the # of i-element subsets of A. Thus,

$$|\mathcal{P}_n(Q)| = (\text{\# of } n\text{-element subsets of } Q) = \binom{x+y}{n} \qquad (3)$$

(by the combinatorial interpretation of BCs, since Q is an $(x+y)$-element set).

Furthermore, for any $k \in \{0, 1, \ldots, n\}$, let

$$\mathcal{P}_{n,k}(Q) := \{n\text{-element subsets of } Q \text{ with exactly } k \text{ positive elements}\}.$$

Then, the sum rule yields

$$|\mathcal{P}_n(Q)| = \sum_{k=0}^{n} |\mathcal{P}_{n,k}(Q)|, \qquad (4)$$

since any n-element subset of Q has exactly k positive elements for a unique $k \in \{0, 1, \ldots, n\}$.

However, for each $k \in \{0, 1, \ldots, n\}$, there is a bijection from $\mathcal{P}_{n,k}(Q)$ to the Cartesian product

$$\begin{aligned}
&\mathcal{P}_k(\{1, 2, \ldots, x\}) \times \mathcal{P}_{n-k}(\{-1, -2, \ldots, -y\}) \\
&= \{\text{pairs } (X, Y), \text{ where } X \text{ is a } k\text{-element subset of } \{1, 2, \ldots, x\} \\
&\qquad \text{and } Y \text{ is an } (n-k)\text{-element subset of } \{-1, -2, \ldots, -y\}\};
\end{aligned}$$

this bijection sends each subset $S \in \mathcal{P}_{n,k}(Q)$ to the pair

$$\begin{aligned}
(S_+, S_-), \text{ where } S_+ &:= \{\text{positive elements of } S\} = S \cap \{1, 2, \ldots, x\} \\
\text{and } S_- &:= \{\text{negative elements of } S\} = S \cap \{-1, -2, \ldots, -y\}.
\end{aligned}$$

The bijection principle therefore yields

$$\begin{aligned}
|\mathcal{P}_{n,k}(Q)| &= |\mathcal{P}_k(\{1, 2, \ldots, x\}) \times \mathcal{P}_{n-k}(\{-1, -2, \ldots, -y\})| \\
&= \underbrace{|\mathcal{P}_k(\{1, 2, \ldots, x\})|}_{\substack{=\binom{x}{k} \\ \text{(by the combinatorial} \\ \text{interpretation of BCs)}}} \cdot \underbrace{|\mathcal{P}_{n-k}(\{-1, -2, \ldots, -y\})|}_{\substack{=\binom{y}{n-k} \\ \text{(by the combinatorial} \\ \text{interpretation of BCs)}}} \qquad \text{(by the product rule)} \\
&= \binom{x}{k}\binom{y}{n-k}.
\end{aligned}$$

Thus, (4) can be rewritten as

$$|\mathcal{P}_n(Q)| = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}.$$

Comparing this with (3), we obtain

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}.$$

This proves Theorem 2.6.2 for all $x, y \in \mathbb{N}$ again.

2.6.3. The upside-down Chu–Vandermonde identity

Recall the upside-down Chu–Vandermonde identity (Lecture 15, Proposition 2.3.6):

Proposition 2.6.4 (upside-down Chu–Vandermonde identity)**.** Let $n, x, y \in \mathbb{N}$. Then,

$$\binom{n+1}{x+y+1} = \sum_{k=0}^{n} \binom{k}{x} \binom{n-k}{y}.$$

Again, let us prove this combinatorially.

Combinatorial proof of Proposition 2.6.4. We proceed by double counting.

How many $(x+y+1)$-element subsets does the set $[n+1]$ have? Again, there are two ways to answer this question:

1st way: There are $\binom{n+1}{x+y+1}$ of them, by the combinatorial interpretation of BCs.

2nd way: We can construct an $(x+y+1)$-element subset S of $[n+1]$ by the following procedure:

- First, we choose the $(x+1)$-st smallest element of S. Let k be this element. Note that $k \in [n+1] = \{1, 2, \ldots, n+1\}$.

  Note that the set S will have to contain exactly x elements smaller than k (since k is to be the $(x+1)$-st smallest element of S) and exactly y elements larger than k (since S must have $x+y+1$ elements in total, but k is its $(x+1)$-st smallest element).

- Next, we choose the x elements of S that are smaller than k. These x elements must be chosen from $\{1, 2, \ldots, k-1\}$. So there are $\binom{k-1}{x}$ options for them, since we are choosing an x-element subset of $\{1, 2, \ldots, k-1\}$.

- Finally, we choose the y elements of S that are larger than k. These y elements must be chosen from $\{k+1, k+2, \ldots, n+1\}$. So there are $\binom{n-k+1}{y}$ options for them, since we are choosing a y-element subset of $\{k+1, k+2, \ldots, n+1\}$.

Altogether, the # of ways to perform this construction is

$$\begin{aligned}\sum_{k=1}^{n+1} \binom{k-1}{x} \binom{n-k+1}{y} &= \sum_{k=1}^{n+1} \binom{k-1}{x} \binom{n-(k-1)}{y} \\ &= \sum_{k=0}^{n} \binom{k}{x} \binom{n-k}{y}\end{aligned}$$

(here, we have substituted k for $k-1$ in the sum).

Comparing the results of both ways, we find

$$\binom{n+1}{x+y+1} = \sum_{k=0}^{n} \binom{k}{x} \binom{n-k}{y},$$

which is precisely the claim of Proposition 2.6.4. □

Remark 2.6.5. Applying Proposition 2.6.4 to $y = 0$, we obtain

$$\binom{n+1}{x+1} = \sum_{k=0}^{n} \binom{k}{x}$$

(because $\binom{n-k}{0} = 1$). This is precisely the hockey-stick identity (part of Lecture 7, Theorem 1.3.24). Thus, we have given a combinatorial proof of the hockey-stick identity.

2.6.4. Expanding the product of two BCs

Here is another binomial identity (one we haven't seen so far):

Theorem 2.6.6. Let $a, b \in \mathbb{N}$ and $x \in \mathbb{R}$. Then,

$$\binom{x}{a}\binom{x}{b} = \sum_{i=0}^{a+b} \binom{i}{a} \binom{a}{a+b-i} \binom{x}{i}.$$

Example 2.6.7. For $a = 2$ and $b = 3$, the claim of Theorem 2.6.6 says that

$$\begin{aligned}
\binom{x}{2}\binom{x}{3} &= \sum_{i=0}^{2+3} \binom{i}{2} \binom{2}{2+3-i} \binom{x}{i} \\
&= \underbrace{\binom{0}{2}}_{=0} \binom{2}{5} \binom{x}{0} + \underbrace{\binom{1}{2}}_{=0} \binom{2}{4} \binom{x}{1} + \binom{2}{2} \underbrace{\binom{2}{3}}_{=0} \binom{x}{2} \\
&\qquad + \underbrace{\binom{3}{2}}_{=3} \underbrace{\binom{2}{2}}_{=1} \binom{x}{3} + \underbrace{\binom{4}{2}}_{=6} \underbrace{\binom{2}{1}}_{=2} \binom{x}{4} + \underbrace{\binom{5}{2}}_{=10} \underbrace{\binom{2}{0}}_{=1} \binom{x}{5} \\
&= 3\binom{x}{3} + 12\binom{x}{4} + 10\binom{x}{5}.
\end{aligned}$$

An algebraic proof of Theorem 2.6.6 appears in [Grinbe15, Proposition 3.37][3]. Here is a combinatorial proof:

[3] More precisely, [Grinbe15, Proposition 3.37] is Theorem 2.6.6 with $\mathbb{R}$ replaced by $\mathbb{Q}$, but the difference between $\mathbb{R}$ and $\mathbb{Q}$ does not matter to the proof.

Combinatorial proof of Theorem 2.6.6. First of all, we WLOG assume that $x \in \mathbb{N}$, because if we can prove the claim for all $x \in \mathbb{N}$, then we will automatically conclude (using the polynomial identity trick, applied to the polynomials $P = \binom{X}{a}\binom{X}{b}$ and $Q = \sum_{i=0}^{a+b} \binom{i}{a}\binom{a}{a+b-i}\binom{X}{i}$) that it also holds for all $x \in \mathbb{R}$.

Thus, $x \in \mathbb{N}$. Let X be the set $[x] = \{1, 2, \ldots, x\}$.

Let us count the pairs (A, B) that consist of an a-element subset A of X and a b-element subset B of X. We shall refer to such pairs (A, B) as "**ab-pairs**"[4]. We count these ab-pairs (A, B) in two ways:

1st way: The # of ab-pairs is $\binom{x}{a}\binom{x}{b}$, since there are $\binom{x}{a}$ ways to choose A and $\binom{x}{b}$ ways to choose B.

2nd way: Here is a more intricate way to construct an ab-pair (A, B):

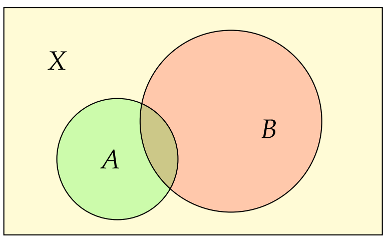
- First, we choose the size $i := |A \cup B|$. This size i is an element of $\{0, 1, \ldots, a+b\}$ (since we must have $i = |A \cup B| \leq |A| + |B| = a + b$).
- Next, we choose the set $A \cup B$ itself. This must be an i-element subset of X, so it can be chosen in $\binom{x}{i}$ many ways.
- Next, we choose the set A. This must be an a-element subset of the union $A \cup B$ (which is already fixed), so it can be chosen in $\binom{i}{a}$ many ways (since $A \cup B$ is an i-element set).
- Finally, we choose the set $A \cap B$. This must be a subset of A, and its size must be $a + b - i$. Why? Well, for any two finite sets U and V, we have $|U \cup V| + |U \cap V| = |U| + |V|$ (this is a nice and easy identity to check[5]), so that

  $$|U \cap V| = |U| + |V| - |U \cup V|.$$

  Applying this to $U = A$ and $V = B$, we obtain

  $$|A \cap B| = \underbrace{|A|}_{=a} + \underbrace{|B|}_{=b} - \underbrace{|A \cup B|}_{=i} = a + b - i.$$

[4]The following illustration of an ab-pair might be useful:



[5]and will be greatly generalized in the next lecture!

Thus, $A \cap B$ should be an $(a+b-i)$-element subset of A. There are $\dbinom{a}{a+b-i}$ many options for choosing such a subset.

- Having chosen $A \cup B$, A and $A \cap B$, we get B for free using the formula

$$B = ((A \cup B) \setminus A) \cup (A \cap B)$$

(which is true for any two sets A and B and follows easily by looking at Venn diagrams).

I claim that the pair (A, B) we have constructed is an ab-pair. Proving this is slightly tricky because our notation is a bit too suggestive: We have chosen three sets $A \cup B$, A and $A \cap B$ and only then defined the set B; thus, it is not obvious that the previously chosen set $A \cup B$ really is the union of our two sets A and B, nor is it clear that the set $A \cap B$ really is the intersection of our two sets A and B. The safest way around this minefield is to rename the sets $A \cup B$ and $A \cap B$ that were chosen in our above steps as C and D, respectively. Thus, C is an i-element subset of X, and A is an a-element subset of C, and D is an $(a+b-i)$-element subset of A. The definition of B can then be rewritten as $B = (C \setminus A) \cup D$. We must now prove that $A \cup B = C$ and $A \cap B = D$. This is straightforward set theory:

$$A \cup \underbrace{B}_{=(C \setminus A) \cup D} = \underbrace{A \cup (C \setminus A)}_{\substack{=C \\ (\text{since } A \subseteq C)}} \cup D = C \cup D = C \qquad (\text{since } D \subseteq A \subseteq C)$$

and

$$\begin{aligned} A \cap \underbrace{B}_{=(C \setminus A) \cup D} &= A \cap ((C \setminus A) \cup D) \\ &= \underbrace{(A \cap (C \setminus A))}_{\substack{=\varnothing \\ (\text{since } A \text{ and } C \setminus A \text{ are disjoint})}} \cup (A \cap D) \\ &\qquad \left(\begin{array}{c} \text{by the distributive law} \\ U \cap (V \cup W) = (U \cap V) \cup (U \cap W), \\ \text{which holds for any three sets } U, V, W \end{array} \right) \\ &= \varnothing \cup (A \cap D) = A \cap D = D \qquad (\text{since } D \subseteq A). \end{aligned}$$

Finally, we need to prove that B is a b-element set. This, too, is easy: The set $C \setminus A$ is disjoint from A (clearly), and thus also disjoint from D (since $D \subseteq A$). Hence, by the sum rule, we have $|(C \setminus A) \cup D| = |C \setminus A| + |D|$.

However, $B = (C \setminus A) \cup D$, so that

$$|B| = |(C \setminus A) \cup D| = \underbrace{|C \setminus A|}_{\substack{=|C|-|A| \\ \text{(since } A\subseteq C)}} + \underbrace{|D|}_{=a+b-i} = \underbrace{|C|}_{=i} - \underbrace{|A|}_{=a} + a + b - i$$
$$= i - a + a + b - i = b.$$

This shows that B is a b-element set. Thus, we have finally shown that (A, B) really is an ab-pair.

Altogether, by the sum rule and the dependent product rule, we see that the # of ways to perform this construction is

$$\sum_{i=0}^{a+b} \binom{x}{i} \binom{i}{a} \binom{a}{a+b-i}.$$

Thus, the # of ab-pairs is

$$\sum_{i=0}^{a+b} \binom{x}{i} \binom{i}{a} \binom{a}{a+b-i}$$

(since any ab-pair can clearly be obtained by our construction).

Comparing our two formulas for the # of ab-pairs, we obtain

$$\binom{x}{a}\binom{x}{b} = \sum_{i=0}^{a+b} \binom{x}{i} \binom{i}{a} \binom{a}{a+b-i} = \sum_{i=0}^{a+b} \binom{i}{a} \binom{a}{a+b-i} \binom{x}{i}.$$

This proves Theorem 2.6.6. □

2.6.5. The combinatorial interpretation of the BCs

Let us now go back to the basics. The combinatorial interpretation of the BCs (Lecture 6, Theorem 1.3.10) says the following:

> **Theorem 2.6.8.** Let $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Let S be an n-element set. Then,
>
> $$(\text{\# of } k\text{-element subsets of } S) = \binom{n}{k}.$$

Back in Lecture 6, we proved this by induction. This was already a rather combinatorial proof (as it used bijections), but let us give a new, "more combinatorial" proof now:

New proof of Theorem 2.6.8. We WLOG assume that $k \in \mathbb{N}$ (since otherwise, our claim is just saying that $0 = 0$).

Thus, by the definition of BCs, we have

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k^{\underline{k}}},$$

where

$$n^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1)$$

is the falling factorial defined in Definition 2.4.2 (Lecture 16). Hence, we must prove that

$$(\text{\# of } k\text{-element subsets of } S) = \frac{n^{\underline{k}}}{k^{\underline{k}}}.$$

Equivalently, we must prove that

$$(\text{\# of } k\text{-element subsets of } S) \cdot k^{\underline{k}} = n^{\underline{k}}.$$

We use double counting again. We let $K := [k] = \{1, 2, \ldots, k\}$. What is the # of injective maps from K to S ? We answer this question in two ways:

1st way: Theorem 2.4.4 (from Lecture 16) yields

$$(\text{\# of injective maps from } K \text{ to } S) = n^{\underline{k}} \tag{5}$$

(since K is a k-element set, while S is an n-element set).

2nd way: We recall the notion of the **range** of a map.

If $f : X \to Y$ is any map, then the **range** (aka **image**) of f is defined to be the subset

$$\operatorname{Im} f := \{f(x) \mid x \in X\} = \{\text{all values of } f\}$$

of Y. Thus, in particular, for any map $f : K \to S$, the range $\operatorname{Im} f$ is a subset of S. Moreover, if the map $f : K \to S$ is injective, then its range $\operatorname{Im} f$ is a k-element set[6] and therefore is a k-element subset of S.

So any injective map $f : K \to S$ has a range $\operatorname{Im} f$ that is a k-element subset of S. Thus, by the sum rule,

$$\begin{aligned} &(\text{\# of injective maps from } K \text{ to } S) \\ &= \sum_{R \text{ is a } k\text{-element subset of } S} (\text{\# of injective maps from } K \text{ to } S \text{ whose range is } R). \end{aligned}$$

Let us now take a closer look at the addends on the right hand side.

We fix a k-element subset R of S. Then,

$$\begin{aligned} &(\text{\# of injective maps from } K \text{ to } S \text{ whose range is } R) \\ &= (\text{\# of bijective maps from } K \text{ to } R) \end{aligned}$$

[6] Indeed, if f is injective, then the values $f(1), f(2), \ldots, f(k)$ of f are distinct, so that f has k distinct values, but this means that the set $\operatorname{Im} f$ has k elements.

(since an injective map from K to S whose range is R is essentially the same as a bijective map from K to R [7]). Furthermore, the two sets K and R satisfy $|K| = |R|$ (since $|K| = k$ and $|R| = k$). Thus, the Pigeonhole Principle for Injections (Theorem 2.4.5 **(b)** in Lecture 16) yields that any injective map from K to R is bijective. Since the converse holds as well (every bijective map is automatically injective), we thus conclude that the bijective maps from K to R

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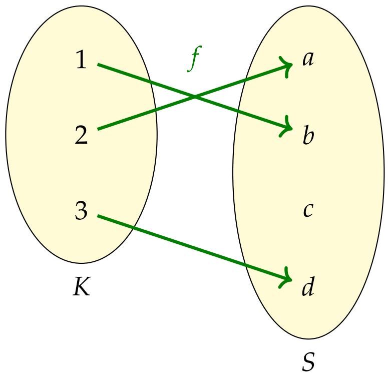
[7] More details for the confused and the pedantic:

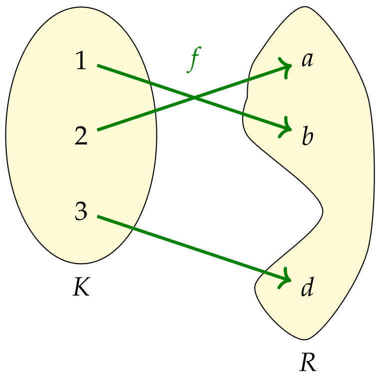
Let f be a map from K to S whose range is R. Then, the values of f belong to R. Thus, f is "essentially" a map from K to R. To be more precise, we can define a map $f' : K \to R$ by letting

$$f'(x) = f(x) \qquad \text{for all } x \in K.$$

For example,

if $f =$ 

(and $R = \{a, b, d\}$),

then $f' =$ 

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Strictly speaking, we should make a distinction between the maps f and f', since they have different targets (S for f and R for f'), but it is clear that they are "essentially the same map" (as they send the same elements to the same values). Note furthermore that the map f' is surjective (since its target R is the range of f, so that every element of R is taken as a value by f and thus by f' as well). If f is injective, then f' is also injective and therefore bijective (since f' is surjective).

Forget that we fixed f. We thus have shown that if f is an injective map from K to S whose range is R, then the corresponding map f' from K to R is bijective. Conversely, any bijective map from K to R can be viewed as an injective map from K to S whose range is R.

are precisely the injective maps from K to R. Therefore,

$$\begin{aligned}&(\text{\# of bijective maps from } K \text{ to } R)\\&= (\text{\# of injective maps from } K \text{ to } R) = k^{\underline{k}}\end{aligned}$$

(by Theorem 2.4.4 in Lecture 16, since $|K| = k$ and $|R| = k$). Altogether, we thus have

$$\begin{aligned}&(\text{\# of injective maps from } K \text{ to } S \text{ whose range is } R)\\&= (\text{\# of bijective maps from } K \text{ to } R)\\&= k^{\underline{k}}.\end{aligned} \tag{6}$$

Forget that we fixed R. We thus have proved (6) for every k-element subset R of S.

Now, we can continue our computation of (# of injective maps from K to S) as follows:

$$\begin{aligned}&(\text{\# of injective maps from } K \text{ to } S)\\&= \sum_{R \text{ is a } k\text{-element subset of } S} \underbrace{(\text{\# of injective maps from } K \text{ to } S \text{ whose range is } R)}_{\substack{= k^{\underline{k}} \\ (\text{by (6)})}}\\&= \sum_{R \text{ is a } k\text{-element subset of } S} k^{\underline{k}}\\&= (\text{\# of } k\text{-element subsets of } S) \cdot k^{\underline{k}}.\end{aligned}$$

Comparing this with (5), we obtain

$$(\text{\# of } k\text{-element subsets of } S) \cdot k^{\underline{k}} = n^{\underline{k}}.$$

As explained above, this proves Theorem 2.6.8. □

2.6.6. An identity with fractions

Here is another apocryphal binomial identity:

Theorem 2.6.9. Let $n, m \in \mathbb{N}$. Then,

$$n = \sum_{k=0}^{m-1} \binom{n-1}{k} \frac{(k+1)!}{n^k} + n \binom{n-1}{m} \frac{m!}{n^m}.$$

Proof. By multiplying both sides with n^{m-1}, we can rewrite this identity as

$$n^m = \sum_{k=0}^{m-1} \binom{n-1}{k} (k+1)! n^{m-1-k} + \binom{n-1}{m} m!. \tag{7}$$

We shall prove this by double counting. Specifically, we shall double-count the # of m-tuples $(a_1, a_2, \ldots, a_m) \in [n]^m$.

1st way: Clearly, the # of such m-tuples is n^m.

2nd way: We define the **wit** of an m-tuple $(a_1, a_2, \ldots, a_m)$ to be the largest $i \in \{0, 1, \ldots, m\}$ such that the numbers $a_1, a_2, \ldots, a_i$ are distinct.

For example,

- the 5-tuple $(1, 4, 2, 2, 1)$ has wit 3 (since $1, 4, 2$ are distinct, but $1, 4, 2, 2$ are not).
- the 5-tuple $(4, 5, 6, 5, 4)$ has wit 3.
- the 5-tuple $(2, 4, 6, 8, 9)$ has wit 5.

Note that the wit is always an element of $[m]$ unless $m = 0$. We WLOG assume that $m \neq 0$ (since (7) is easy to prove by hand for $m = 0$). Thus, the wit is always an element of $[m]$.

Hence, the sum rule yields

$$\begin{aligned} &\left(\# \text{ of } m\text{-tuples } (a_1, a_2, \ldots, a_m) \in [n]^m\right) \\ &= \sum_{i=1}^{m} \left(\# \text{ of } m\text{-tuples } (a_1, a_2, \ldots, a_m) \in [n]^m \text{ with wit } i\right). \end{aligned} \tag{8}$$

So let us count the m-tuples with a given wit.

Fix $i \in [m-1]$. How many m-tuples $(a_1, a_2, \ldots, a_m) \in [n]^m$ have wit i ? An m-tuple $(a_1, a_2, \ldots, a_m) \in [n]^m$ has wit i if and only if

- its first i entries $a_1, a_2, \ldots, a_i$ are distinct,
- its next entry a_{i+1} repeats one of these i entries,
- and all its remaining $m - i - 1$ entries $a_{i+2}, a_{i+3}, \ldots, a_m$ are arbitrary.

Thus, we can construct an m-tuple $(a_1, a_2, \ldots, a_m) \in [n]^m$ with wit i via the following procedure:

- We choose its first i entries $a_1, a_2, \ldots, a_i$. The # of options for this is $n(n-1)(n-2)\cdots(n-i+1)$ (since each of these i entries has to be distinct from all the preceding entries, which are furthermore distinct among themselves by construction).
- Next, we choose the $(i+1)$-st entry a_{i+1}. The # options for this is i (since it has to repeat one of the previous i entries, which are furthermore distinct by construction).
- Finally, we choose the remaining $m - i - 1$ entries $a_{i+2}, a_{i+3}, \ldots, a_m$. The # of options for this is n^{m-i-1} (since they are arbitrarily chosen from $[n]$).

By the dependent product rule, the total # of ways to perform this procedure is

$$\begin{aligned}
&\underbrace{n\left(n-1\right)\left(n-2\right)\cdots\left(n-i+1\right)}_{=\binom{n}{i}\cdot i!}\cdot i\cdot n^{m-i-1} \\
&= \underbrace{\binom{n}{i}}_{\substack{=\frac{n}{i}\binom{n-1}{i-1} \\ \text{(by the absorption formula} \\ \text{(Lecture 8, Proposition 1.3.29))}}}\cdot i!\cdot i\cdot n^{m-i-1} \\
&= \frac{n}{i}\binom{n-1}{i-1}\cdot i!\cdot i\cdot n^{m-i-1} \\
&= \binom{n-1}{i-1}\cdot i!\cdot n^{m-i}.
\end{aligned}$$

So this must be the # of m-tuples $(a_1, a_2, \ldots, a_m) \in [n]^m$ with wit i.

Forget that we fixed i. We have thus proved that

$$\begin{aligned}
&\left(\text{\# of } m\text{-tuples } (a_1, a_2, \ldots, a_m) \in [n]^m \text{ with wit } i\right) \\
&= \binom{n-1}{i-1}\cdot i!\cdot n^{m-i}
\end{aligned} \tag{9}$$

for each $i \in [m-1]$.

When $i = m$, a similar argument gives us

$$\begin{aligned}
&\left(\text{\# of } m\text{-tuples } (a_1, a_2, \ldots, a_m) \in [n]^m \text{ with wit } m\right) \\
&= n\left(n-1\right)\left(n-2\right)\cdots\left(n-m+1\right) = \binom{n}{m}\cdot m!.
\end{aligned} \tag{10}$$

Now, (8) becomes

$$\begin{aligned}
&\left(\#\text{ of } m\text{-tuples } (a_1,a_2,\ldots,a_m)\in[n]^m\right)\\
&=\sum_{i=1}^{m}\left(\#\text{ of } m\text{-tuples } (a_1,a_2,\ldots,a_m)\in[n]^m \text{ with wit } i\right)\\
&=\sum_{i=1}^{m-1}\underbrace{\left(\#\text{ of } m\text{-tuples } (a_1,a_2,\ldots,a_m)\in[n]^m \text{ with wit } i\right)}_{\substack{=\binom{n-1}{i-1}\cdot i!\cdot n^{m-i}\\ \text{(by (9))}}}\\
&\qquad+\underbrace{\left(\#\text{ of } m\text{-tuples } (a_1,a_2,\ldots,a_m)\in[n]^m \text{ with wit } m\right)}_{\substack{=\binom{n}{m}\cdot m!\\ \text{(by (10))}}}\\
&=\sum_{i=1}^{m-1}\binom{n-1}{i-1}\cdot i!\cdot n^{m-i}+\binom{n}{m}\cdot m!\\
&=\sum_{k=0}^{m-2}\binom{n-1}{k}\cdot(k+1)!\cdot\underbrace{n^{m-(k+1)}}_{=n^{m-1-k}}+\underbrace{\binom{n}{m}}_{\substack{=\binom{n-1}{m-1}+\binom{n-1}{m}\\ \text{(by Pascal's recurrence)}}}\cdot m!\\
&\qquad\left(\text{here, we substituted } k+1 \text{ for } i \text{ in the sum}\right)\\
&=\sum_{k=0}^{m-2}\binom{n-1}{k}\cdot(k+1)!\cdot n^{m-1-k}+\left(\binom{n-1}{m-1}+\binom{n-1}{m}\right)\cdot m!\\
&=\underbrace{\sum_{k=0}^{m-2}\binom{n-1}{k}(k+1)!n^{m-1-k}+\binom{n-1}{m-1}m!}_{\substack{=\sum_{k=0}^{m-1}\binom{n-1}{k}(k+1)!n^{m-1-k}\\ \text{(since the extra addend }\binom{n-1}{m-1}m!\\ \text{is precisely the ``missing'' } k=m-1 \text{ addend}\\ \text{of the sum }\sum_{k=0}^{m-2}\binom{n-1}{k}(k+1)!n^{m-1-k})}}+\binom{n-1}{m}m!\\
&=\sum_{k=0}^{m-1}\binom{n-1}{k}(k+1)!n^{m-1-k}+\binom{n-1}{m}m!.
\end{aligned}$$

Comparing this with

$$\left(\#\text{ of } m\text{-tuples } (a_1,a_2,\ldots,a_m)\in[n]^m\right)=n^m,$$

we obtain

$$n^m = \sum_{k=0}^{m-1} \binom{n-1}{k} (k+1)! n^{m-1-k} + \binom{n-1}{m} m!.$$

This proves (7) and thus Theorem 2.6.9. □

Exercise 1. Prove Theorem 2.6.9 algebraically, by induction on m.

References

[BenQui03] Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, Dolciani Mathematical Expositions **27**, The Mathematical Association of America, 2003.

[Grinbe15] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 15 September 2022.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2022-09-15c` .

[Loehr11] Nicholas A. Loehr, *Bijective Combinatorics*, Chapman & Hall/CRC 2011.