# Math 222 Fall 2022, Lecture 10: Introduction

**website:** `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

# 1. Introduction (cont'd)

## 1.4. Counting subsets (cont'd)

### 1.4.6. Counting subsets with $a$ odd and $b$ even elements (cont'd)

Last time, I stated the following proposition:

**Proposition 1.4.11.** Let $n \in \mathbb{N}$ be even. Let $a, b \in \mathbb{N}$. Then,

$$\begin{aligned}&(\text{\# of subsets of } [n] \text{ that contain } a \text{ even elements and } b \text{ odd elements}) \\ &= \dbinom{n/2}{a}\dbinom{n/2}{b}.\end{aligned}$$

(Here, "$a$ even elements" means "exactly $a$ even elements", and "$b$ odd elements" means "exactly $b$ odd elements".)

Let us now prove it:

*Proof.* (See the proof of Proposition 1.4.14 in the 2019 notes for details.)

We will give an informal proof and a formal one. More precisely, we will give one proof in an informal writeup and in a formal one. (You don't need to spell out your proofs formally on the homework, but you should know how this can be done if so desired.)

*Informal proof:* To construct a subset of $[n]$ that contains $a$ even elements and $b$ odd elements, we just need to choose $a$ elements from the set $E := \{2, 4, 6, \ldots, n\}$ and to choose $b$ elements from the set $O := \{1, 3, 5, \ldots, n-1\}$. We have $\dbinom{n/2}{a}$ options for our first choice (by the combinatorial interpretation of BCs, since $|E| = n/2$) and $\dbinom{n/2}{b}$ options for our second choice (similarly). Since the two choices are independent, we thus have $\dbinom{n/2}{a}\dbinom{n/2}{b}$ possibilities in total.

*Formal proof:* This "independent choice" language we just used is just shorthand for applying the product rule. To state our above argument formally, we introduce the notation $\mathcal{P}_k(S)$ for the set of all $k$-element subsets of a given set $S$. We also let $\mathcal{N}$ denote the set of all subsets of $[n]$ that contain $a$ even elements and $b$ odd elements. We must then prove that $|\mathcal{N}| = \dbinom{n/2}{a}\dbinom{n/2}{b}$.

We set $E := \{2, 4, 6, \ldots, n\}$ and $O := \{1, 3, 5, \ldots, n-1\}$. The map

$$\mathcal{N} \to \mathcal{P}_a(E) \times \mathcal{P}_b(O),$$
$$S \mapsto (S \cap E,\ S \cap O)$$

is a bijection, since the map

$$\mathcal{P}_a(E) \times \mathcal{P}_b(O) \to \mathcal{N},$$
$$(U, V) \mapsto U \cup V$$

is inverse to it. So, by the bijection principle,

$$\begin{aligned}
|\mathcal{N}| &= |\mathcal{P}_a(E) \times \mathcal{P}_b(O)| = |\mathcal{P}_a(E)| \cdot |\mathcal{P}_b(O)| && \text{(by the product rule)} \\
&= \binom{|E|}{a} \cdot \binom{|O|}{b} && \left(\begin{array}{c}\text{since the combinatorial interpretation of} \\ \text{BCs says that } |\mathcal{P}_k(S)| = \dbinom{|S|}{k} \text{ for all } S \text{ and } k\end{array}\right) \\
&= \binom{n/2}{a} \cdot \binom{n/2}{b} && (\text{since } |E| = n/2 \text{ and } |O| = n/2).
\end{aligned}$$

This again proves Proposition 1.4.11. □

A takeaway from the above argument: "Independent choices" correspond to an application of the product rule (and possibly of the bijection principle).

### 1.4.7. The addition formula for Fibonacci numbers

For another example of counting subsets, we shall prove combinatorially the following property of Fibonacci numbers:

**Theorem 1.4.12** (addition formula for Fibonacci numbers)**.** Let $m, n \in \mathbb{N}$. Then, the Fibonacci sequence satisfies

$$f_{m+n+1} = f_m f_n + f_{m+1} f_{n+1}.$$

There are various easy ways to prove this (e.g., by induction[1], or using Binet's formula). But let me show a **combinatorial** proof, to illustrate a strategy.

First, we begin with a simple piece of notation:

**Definition 1.4.13.** Let $a$ and $b$ be two integers. Then, the subset

$$\{a, a+1, \ldots, b\} = \{i \in \mathbb{Z} \mid a \leq i \leq b\} \qquad \text{of } \mathbb{Z}$$

will be denoted by $[a, b]$. This subset is called the **integer interval** from $a$ to $b$. It is empty if $a > b$.

[1] For such a proof, see [20f, Exercise 2.2.3]. (See also [20f, Exercise 4.9.3 and Exercise 4.9.4] for generalizations.)

For example, $[2,5] = \{2,3,4,5\}$. Note that $[k] = [1,k]$ for every $k \in \mathbb{Z}$.
We can now generalize Proposition 1.4.7 from Lecture 9 as follows:

**Proposition 1.4.14.** Let $n \in \{-1,0,1,\ldots\}$ and $a \in \mathbb{Z}$. Then,

$$(\text{\# of lacunar subsets of } [a+1, a+n]) = f_{n+2}.$$

*Proof.* Proposition 1.4.7 from Lecture 9 says that

$$(\text{\# of lacunar subsets of } [n]) = f_{n+2}.$$

However, there is a bijection

$$\{\text{lacunar subsets of } [n]\} \to \{\text{lacunar subsets of } [a+1, a+n]\},$$
$$S \mapsto \{s + a \mid s \in S\}$$

(this is simply shifting a lacunar subset of $[n]$ by $a$ units on the number line, so that it becomes a lacunar subset of $[a+1, a+n]$). Thus, by the bijection principle, we have

$$(\text{\# of lacunar subsets of } [a+1, a+n]) = (\text{\# of lacunar subsets of } [n]) = f_{n+2}.$$

This proves Proposition 1.4.14. □

*Proof of Theorem 1.4.12 (sketched).* This will be a very rough sketch; see the 2019 notes (§1.4.7) for details.

For $m = 0$, the claim is obvious (since $f_0 = 0$ and $f_1 = 1$). So we WLOG assume that $m \geq 1$. Similarly, we WLOG assume that $n \geq 1$. Now, let us count the lacunar subsets of $[m+n-1]$ in two ways:

*First way:* Proposition 1.4.14 (applied to 0 and $m+n-1$ instead of $a$ and $n$) yields

$$(\text{\# of lacunar subsets of } [m+n-1]) = f_{(m+n-1)+2} = f_{m+n+1}.$$

*Second way:* We shall call a subset of $[m+n-1]$

- **red** if it contains $m$, and
- **green** if it does not contain $m$.

Thus, by the sum rule, we have

$$\begin{aligned}
&(\text{\# of lacunar subsets of } [m+n-1]) \\
&= (\text{\# of lacunar red subsets of } [m+n-1]) \\
&\qquad + (\text{\# of lacunar green subsets of } [m+n-1]).
\end{aligned}$$

A lacunar red subset of $[m+n-1]$ must contain $m$, and thus cannot contain any of $m-1$ and $m+1$ (since it is lacunar). Hence, if we remove $m$ from it, then we obtain a union of a lacunar subset of $[m-2]$ with a lacunar subset of $[m+2,\ m+n-1]$. Conversely, if we take a union of any lacunar subset of $[m-2]$ with any lacunar subset of $[m+2,\ m+n-1]$, and if we insert $m$ into this union, then we obtain a lacunar red subset of $[m+n-1]$. Hence, we have a bijection

$$\begin{aligned}&\{\text{lacunar red subsets of } [m+n-1]\}\\ &\to \{\text{lacunar subsets of } [m-2]\} \times \{\text{lacunar subsets of } [m+2,\ m+n-1]\}\end{aligned}$$

(which sends each subset $S$ to the pair $(S\cap[m-2],\ S\cap[m+2,\ m+n-1])$). Hence, by the bijection principle,

$$\begin{aligned}&|\{\text{lacunar red subsets of } [m+n-1]\}|\\ &= |\{\text{lacunar subsets of } [m-2]\} \times \{\text{lacunar subsets of } [m+2,\ m+n-1]\}|\\ &= \underbrace{|\{\text{lacunar subsets of } [m-2]\}|}_{\substack{=f_m\\ \text{(by Proposition 1.4.14,}\\ \text{applied to 0 and } m-2\\ \text{instead of } a \text{ and } n)}} \cdot \underbrace{|\{\text{lacunar subsets of } [m+2,\ m+n-1]\}|}_{\substack{=f_n\\ \text{(by Proposition 1.4.14,}\\ \text{applied to } m+1 \text{ and } n-2\\ \text{instead of } a \text{ and } n)}}\\ &= f_m f_n.\end{aligned}$$

In other words,

$$(\#\text{ of lacunar red subsets of } [m+n-1]) = f_m f_n.$$

A similar argument (see the 2019 notes for details) yields

$$\begin{aligned}&|\{\text{lacunar green subsets of } [m+n-1]\}|\\ &= |\{\text{lacunar subsets of } [m-1]\} \times \{\text{lacunar subsets of } [m+1,\ m+n-1]\}|\\ &= \underbrace{|\{\text{lacunar subsets of } [m-1]\}|}_{=f_{m+1}} \cdot \underbrace{|\{\text{lacunar subsets of } [m+1,\ m+n-1]\}|}_{=f_{n+1}}\\ &= f_{m+1} f_{n+1}.\end{aligned}$$

Altogether,

$$\begin{aligned}&(\#\text{ of lacunar subsets of } [m+n-1])\\ &= \underbrace{(\#\text{ of lacunar red subsets of } [m+n-1])}_{=f_m f_n}\\ &\qquad + \underbrace{(\#\text{ of lacunar green subsets of } [m+n-1])}_{=f_{m+1} f_{n+1}}\\ &= f_m f_n + f_{m+1} f_{n+1}.\end{aligned}$$

So we have solved our counting problem (counting the lacunar subsets of $[m+n-1]$) in two different ways. One way gave us the answer $f_{m+n+1}$, while the other gave us the answer $f_m f_n + f_{m+1} f_{n+1}$. But these two answers must be equal, since they count the same objects. So we get $f_{m+n+1} = f_m f_n + f_{m+1} f_{n+1}$. This proves Theorem 1.4.12. □

The proof we just gave is an example of a **proof by double counting**: We posed ourselves a counting problem; we solved it in two different ways; then we concluded that the two answers must be equal. This is a fairly potent method of proving equalities, even if the equalities do not originally come from combinatorics (as long as both sides of the relevant equality have combinatorial interpretations).

### 1.4.8. More subset counting

The following exercise is solved in the 2019 notes (§1.4.8):

**Exercise 1.** A set $S$ of integers is said to be

- **self-counting** if $|S| \in S$. (For instance, $\{2,3,4\}$ is self-counting, because $3 \in \{2,3,4\}$. But $\{3,4\}$ is not self-counting, because $2 \notin \{3,4\}$).
- **self-starting** if $|S| = \min S$ (that is, the size of $S$ is the smallest element of $S$).
- **self-ending** if $|S| = \max S$ (that is, the size of $S$ is the largest element of $S$).

Let $n \in \mathbb{N}$.
**(a)** For each $k \in [n]$, find the # of self-counting (resp. self-starting, resp. self-ending) $k$-element subsets of $[n]$.
**(b)** Find the # of all self-counting (resp. self-starting, resp. self-ending) subsets of $[n]$.

See the notes for a solution, as well as some other exercises on this topic.

### 1.4.9. Counting subsets containing a given subset

For a given $k \in \mathbb{N}$, how many $k$-element subsets of a given set $N$ contain a given subset $D$ as a subset?

**Proposition 1.4.15.** Let $n \in \mathbb{N}$, $d \in \mathbb{N}$ and $k \in \mathbb{R}$. Let $N$ be an $n$-element set. Let $D$ be a $d$-element subset of $N$. Then,

$$(\text{\# of } k\text{-element subsets } A \text{ of } N \text{ satisfying } D \subseteq A) = \binom{n-d}{k-d}.$$

*Proof.* (See §1.4.9 in the 2019 notes for details.)

*Informal proof:* To build a $k$-element subset $A$ of $N$ satisfying $D \subseteq A$, we start with the $d$ elements of $D$, and we add $k-d$ further elements from the $(n-d)$-element set $N \setminus D$ to them. The latter $k-d$ elements can be chosen arbitrarily. The # of ways to do this is $\binom{n-d}{k-d}$, since it boils down to picking a $(k-d)$-element subset of the $(n-d)$-element set $N \setminus D$.

*Formal proof:* There is a bijection[2]

$$\begin{aligned} \{k\text{-element subsets } A \text{ of } N \text{ satisfying } D \subseteq A\} &\to \mathcal{P}_{k-d}(N \setminus D), \\ A &\mapsto A \setminus D \end{aligned}$$

with inverse map

$$\begin{aligned} \mathcal{P}_{k-d}(N \setminus D) &\to \{k\text{-element subsets } A \text{ of } N \text{ satisfying } D \subseteq A\}, \\ A &\mapsto A \cup D \end{aligned}$$

(yes, it needs to be checked that both of these maps are well-defined and that they are actually inverse to each other; but this is all straightforward set theory). Hence, by the bijection principle,

$$|\{k\text{-element subsets } A \text{ of } N \text{ satisfying } D \subseteq A\}| = |\mathcal{P}_{k-d}(N \setminus D)| = \binom{n-d}{k-d}$$

(by the combinatorial interpretation of BCs, since $|N \setminus D| = n - d$). This proves the proposition. □

## 1.5. Counting tuples and maps

Having counted various kinds of subsets, let us now move on to counting tuples.

### 1.5.1. Tuples

Let us recall the definition of the Cartesian product of sets:

**Definition 1.5.1.** Let $A_1, A_2, \ldots, A_n$ be $n$ sets. Then, their **Cartesian product** $A_1 \times A_2 \times \cdots \times A_n$ is defined to be the set of all $n$-tuples $(a_1, a_2, \ldots, a_n)$, where $a_1 \in A_1$ and $a_2 \in A_2$ and $\cdots$ and $a_n \in A_n$. In other words,

$$\begin{aligned} & A_1 \times A_2 \times \cdots \times A_n \\ &= \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1 \text{ and } a_2 \in A_2 \text{ and } \cdots \text{ and } a_n \in A_n\} \\ &= \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for all } i \in [n]\}. \end{aligned}$$

[2] Again, we let $\mathcal{P}_k(S)$ denote the set of all $k$-element subsets of a given set $S$.

Note that:

- A 2-tuple is the same as a pair (i.e., ordered pair). For example, the Cartesian product $\{1,2\} \times \{5,6,7\}$ contains the pairs (= 2-tuples)

$$(1,5), (1,6), (1,7),$$
$$(2,5), (2,6), (2,7).$$

- A 3-tuple is the same as a triple.

- A 1-tuple consists of a single element (but, to be pedantic, it is not the same as this element). For example, (7) is a 1-tuple of integers. Thus, a Cartesian product of a single set $A_1$ can be viewed as a copy of $A_1$, except that each element is "enclosed in parentheses".

- A 0-tuple has no entries; thus, it looks like this: (). So there is only one 0-tuple.

How do we count these tuples?

**Theorem 1.5.2** (the product rule for $n$ sets)**.** Let $A_1, A_2, \ldots, A_n$ be $n$ finite sets. Then, their Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is again finite and has size

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|.$$

A particularly important example of Cartesian products is when all the factors are the same:

**Definition 1.5.3.** Let $A$ be a set, and let $n \in \mathbb{N}$. Then, the $n$-th **(Cartesian) power** of $A$ is the set

$$A^n := \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}.$$

**Corollary 1.5.4.** Let $A$ be a finite set, and let $n \in \mathbb{N}$. Then, $A^n$ is finite and has size

$$|A^n| = |A|^n.$$

# References

[20f] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 25 December 2021.
`http://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`