Math 222 Fall 2022, Lecture 9: Introduction

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

1. Introduction (cont'd)

1.4. Counting subsets (cont'd)

1.4.2. Lacunar subsets (cont'd)

Last time, we defined lacunar subsets. Recall their definition:

Definition 1.4.2. A set S of integers is said to be **lacunar** if it contains no two consecutive integers (i.e., there is no integer i such that both i and $i+1$ belong to S).

We also answered part **(c)** of the following question:

Question 1.4.3. For given $n,k \in \mathbb{N}$:
(a) How many lacunar subsets does $[n]$ have?
(b) How many k-element lacunar subsets does $[n]$ have?
(c) What is the largest size of a lacunar subset of $[n]$?

We found (and proved) that the largest size of a lacunar subset of $[n]$ is $\lceil n/2 \rceil$. Let us now attack parts **(a)** and **(b)** of this question.

1.4.3. Intermezzo: SageMath

[At this point, I made a little tech demo, showing the use of SageMath via the SageMathCell server (`https://sagecell.sagemath.org/`). See §1.4.3 of the 2019 notes for the details.]

1.4.4. Counting all lacunar subsets

Some experimentation with SageMath suggests the following answer to question **(a)**:[1]

Proposition 1.4.7. Let $n \in \{-1,0,1,\ldots\}$. Then,

$$(\text{\# of lacunar subsets of } [n]) = f_{n+2}.$$

Here, we agree that $[-1] := \varnothing$. More generally, we agree that $[k] := \varnothing$ for any $k \leq 0$.

[1] We extend the range of n from $\mathbb{N}$ to the slightly larger set $\{-1,0,1,\ldots\}$ in order to include a border case that will later appear quite often in applications.

Proof. Forget that we fixed n. For each $n \in \{-1, 0, 1, \ldots\}$, we set

$$\ell_n := (\text{\# of lacunar subsets of } [n]) .$$

Our goal is thus to prove that $\ell_n = f_{n+2}$ for each $n \geq -1$.

We have $\ell_0 = 1$ and $\ell_{-1} = 1$ (since the empty set has only one lacunar subset). We shall now prove the following claaim:

> *Claim 1:* We have $\ell_n = \ell_{n-1} + \ell_{n-2}$ for each $n \geq 1$.

Once this claim is proved, a straightforward strong induction will yield our goal (that $\ell_n = f_{n+2}$ for all $n \geq -1$). So we focus on proving Claim 1.

Proof of Claim 1: Let $n \geq 1$ be an integer. We shall call a subset of $[n]$

- **red** if it contains n, and
- **green** if it does not contain n.

Thus, each subset of $[n]$ is either red or green (but not both). Hence, by the sum rule,

$$\begin{aligned}&(\text{\# of lacunar subsets of } [n]) \\ &= (\text{\# of red lacunar subsets of } [n]) + (\text{\# of green lacunar subsets of } [n]) .\end{aligned}$$

How can we compute the right hand side?

Start with the green lacunar subsets of $[n]$. They are just the lacunar subsets of $[n-1]$. So their number is ℓ_{n-1}. In other words,

$$(\text{\# of green lacunar subsets of } [n]) = \ell_{n-1}.$$

Now to the red lacunar subsets of $[n]$. Inspired by what we did previously, we could try to set up a bijection between $\{\text{red lacunar subsets of } [n]\}$ and $\{\text{lacunar subsets of } [n-1]\}$, by removing n from the set (or inserting n into the set). But this does not work, since inserting n into a lacunar subset of $[n-1]$ will not always yield a lacunar set.

However, we can fix this. If we remove n from a red lacunar subset of $[n]$, then the result is not just a lacunar subset of $[n-1]$, but actually a lacunar subset of $[n-2]$ (since the presence of n in the red subset rules out the presence of $n-1$ by lacunarity). Conversely, inserting n into a lacunar subset of $[n-2]$ always results in a red lacunar subset of $[n]$ (since the subset does not contain $n-1$, and thus the insertion of n will not break its lacunarity). Thus, we obtain a bijection

$$\begin{aligned}\{\text{red lacunar subsets of } [n]\} &\to \{\text{lacunar subsets of } [n-2]\} , \\ S &\mapsto S \setminus \{n\}\end{aligned}$$

with inverse map

$$\begin{aligned}\{\text{lacunar subsets of } [n-2]\} &\to \{\text{red lacunar subsets of } [n]\}, \\ S &\mapsto S \cup \{n\}.\end{aligned}$$

The bijection principle therefore yields

$$(\text{\# of red lacunar subsets of } [n]) = (\text{\# of lacunar subsets of } [n-2]) = \ell_{n-2}.$$

Altogether,

$$\begin{aligned}\ell_n &= (\text{\# of lacunar subsets of } [n]) \\ &= \underbrace{(\text{\# of red lacunar subsets of } [n])}_{=\ell_{n-2}} + \underbrace{(\text{\# of green lacunar subsets of } [n])}_{=\ell_{n-1}} \\ &= \ell_{n-2} + \ell_{n-1} = \ell_{n-1} + \ell_{n-2}.\end{aligned}$$

This proves Claim 1.

As explained above, Proposition 1.4.7 now follows by strong induction. □

There is also a second proof of Proposition 1.4.7, which avoids induction but instead sets up a bijection between the lacunar subsets of $[n]$ and the domino tilings of $R_{n+1,2}$. Thus, by the bijection principle, the # of the lacunar subsets equals the # of domino tilings, which we have already computed (it is f_{n+2}). Details of this argument are found in the 2019 notes (Proposition 1.4.9, second proof).

Either way, part **(a)** of the question is answered.

1.4.5. Counting k-element lacunar subsets

Now to part **(b)**:

Proposition 1.4.8. Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ be such that $k \leq n+1$. Then,

$$(\text{\# of } k\text{-element lacunar subsets of } [n]) = \binom{n+1-k}{k}.$$

One way to prove this proposition is by imitating the above inductive proof of Proposition 1.4.7 (or the similar inductive proof of the fact that the # of k-element subsets of $[n]$ is $\dbinom{n}{k}$): Again, a subset of $[n]$ is called red or green if it contains or doesn't contain n. Having already done such arguments twice, we won't delve into the details here. Pascal's recurrence saves the day.

However, it is also nice to have a more "inspired" proof. After all, the binomial coefficient $\dbinom{n+1-k}{k}$ shouldn't come out for no reason! It counts the k-element subsets of $[n+1-k]$. So it would be reasonable to expect a bijection

$$\begin{aligned}&\text{from } \{k\text{-element lacunar subsets of } [n]\} \\ &\text{to } \{k\text{-element subsets of } [n+1-k]\}.\end{aligned}$$

Such a bijection can indeed be found. Its construction requires a basic fact that I will not prove:

Proposition 1.4.9. Let $m \in \mathbb{N}$. Let S be an m-element set of integers. Then, there exists a unique m-tuple $(s_1, s_2, \ldots, s_m)$ of integers satisfying $\{s_1, s_2, \ldots, s_m\} = S$ and $s_1 < s_2 < \cdots < s_m$.

This is just saying that if you have an m-element set of integers, then there is a unique way to list the elements of this set in (strictly) increasing order. Formally speaking, this needs proof, but this is so basic I will not prove it here (a reference can be found in §1.4.5 of the 2019 notes).

As a consequence of Proposition 1.4.9, any m-element set of integers can be uniquely written in the form $\{s_1 < s_2 < \cdots < s_m\}$, where the $<$-signs mean that the elements are listed in strictly increasing order (for example, the set $\{2, 4, 5\}$ can be written as $\{2 < 4 < 5\}$, not as $\{4 < 2 < 5\}$).

Proof of Proposition 1.4.8. If $\{s_1 < s_2 < s_3 < \cdots < s_k\}$ is a k-element lacunar subset of $[n]$, then we can move its elements closer to each other (letting s_1 stay put, while s_2 moves one step left, while s_3 moves two steps left, and so on), and the resulting k numbers $s_1,\ s_2 - 1,\ s_3 - 2,\ \ldots,\ s_k - (k-1)$ will still be distinct and listed in increasing order (because the lacunarity of $\{s_1 < s_2 < s_3 < \cdots < s_k\}$ ensures that any two of the numbers $s_1, s_2, \ldots, s_k$ have a distance of at least 2 between them). Thus, if $\{s_1 < s_2 < s_3 < \cdots < s_k\}$ is a k-element lacunar subset of $[n]$, then $\{s_1 < s_2 - 1 < s_3 - 2 < \cdots < s_k - (k-1)\}$ is a well-defined k-element set, and in fact a subset of $[n+1-k]$ (since $s_k \leq n$ entails $s_k - (k-1) \leq n - (k-1) = n+1-k$).

Hence, we can define a map

$$\begin{aligned}&\{k\text{-element lacunar subsets of } [n]\} \to \{k\text{-element subsets of } [n+1-k]\}, \\ &\qquad \{s_1 < s_2 < s_3 < \cdots < s_k\} \mapsto \{s_1 < s_2 - 1 < s_3 - 2 < \cdots < s_k - (k-1)\}.\end{aligned}$$

Conversely, we define a map

$$\begin{aligned}&\{k\text{-element subsets of } [n+1-k]\} \to \{k\text{-element lacunar subsets of } [n]\}, \\ &\qquad \{s_1 < s_2 < s_3 < \cdots < s_k\} \mapsto \{s_1 < s_2 + 1 < s_3 + 2 < \cdots < s_k + (k-1)\}\end{aligned}$$

(this produces a lacunar subset, since we added buffer space between every two consecutive elements, and furthermore it will be a subset of $[n]$ because

$s_k \leq n+1-k$ entails $s_k + (k-1) \leq (n+1-k) + (k-1) = n$). These two maps are mutually inverse, and thus are bijections. Hence, by the bijection principle, we have

$$\begin{aligned} &(\text{\# of } k\text{-element lacunar subsets of } [n]) \\ &= (\text{\# of } k\text{-element subsets of } [n+1-k]) = \binom{n+1-k}{k} \end{aligned}$$

(by the combinatorial interpretation of BCs, since $n+1-k \geq 0$). Proposition 1.4.8 is proved. □

Keep the above bijection in mind – it is useful in many other places!

Now we have solved all three parts of our counting question. As a consequence, we can easily prove a binomial identity we left unproved in Lecture 7:

Proposition 1.4.10. Let $n \in \mathbb{N}$. Then, the Fibonacci number f_{n+1} is

$$f_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k} = \binom{n-0}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-n}{n}.$$

Proof. Let $m := n-1 \in \{-1, 0, 1, \ldots\}$. Then,

$$\begin{aligned} (\text{\# of lacunar subsets of } [m]) &= f_{m+2} && (\text{by Proposition 1.4.7}) \\ &= f_{n+1} && (\text{since } m = n-1 \text{ entails } m+2 = n+1). \end{aligned}$$

Thus,

$$\begin{aligned} f_{n+1} &= (\text{\# of lacunar subsets of } [m]) \\ &= \sum_{k=0}^{n} \underbrace{(\text{\# of } k\text{-element lacunar subsets of } [m])}_{\substack{=\binom{m+1-k}{k} \\ \text{(by Proposition 1.4.8)}}} \\ &\qquad \left(\begin{array}{c} \text{by the sum rule, since it is easy to see} \\ \text{that each lacunar subset of } [m] \text{ has size} \\ \text{between 0 and } n \text{ (actually between 0 and } m\text{,} \\ \text{but it is convenient for us to extend this} \\ \text{summation to } m+1=n\text{)} \end{array}\right) \\ &= \sum_{k=0}^{n} \binom{m+1-k}{k} = \sum_{k=0}^{n} \binom{n-k}{k} \qquad (\text{since } m+1=n) \\ &= \binom{n-0}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-n}{n}. \end{aligned}$$

So Proposition 1.4.10 is proved. □

1.4.6. Counting subsets with a odd and b even elements

Here is another instance of counting subsets with a given property:

Proposition 1.4.11. Let $n \in \mathbb{N}$ be even. Let $a, b \in \mathbb{N}$. Then,

$$\begin{aligned}&(\#\text{ of subsets of } [n] \text{ that contain } a \text{ even elements and } b \text{ odd elements})\\&= \binom{n/2}{a}\binom{n/2}{b}.\end{aligned}$$

(Here, "a even elements" means "exactly a even elements", and "b odd elements" means "exactly b odd elements".)

Next time, we'll prove this. Meanwhile, think about why this is true!