Math 222 Fall 2022, Lecture 8: Introduction

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

1. Introduction (cont'd)

1.3. Factorials and binomial coefficients (cont'd)

1.3.7. Other properties of binomial coefficients (cont'd)

Here is another curious property of the n-th row of Pascal's triangle:

Proposition 1.3.27. Let n be a positive integer. Then,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$

Proof. Let

$$e := \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots \qquad \text{and}$$
$$o := \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots .$$

(The letters stand for "even" and "odd"; no relation to 2.718 . . . and 0.) Then,

$$\begin{aligned} e + o &= \left(\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots\right) + \left(\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots\right) \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots = \sum_{k\in\mathbb{N}} \binom{n}{k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \qquad \left(\text{since } \binom{n}{k} = 0 \text{ for all } k > n\right) \\ &= 2^n \qquad (\text{by Corollary 1.3.22 in Lecture 7}) \end{aligned}$$

and

$$\begin{aligned} e - o &= \left(\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots\right) - \left(\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots\right) \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \pm \cdots = \sum_{k\in\mathbb{N}} (-1)^k \binom{n}{k} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \qquad \left(\text{since } \binom{n}{k} = 0 \text{ for all } k > n\right) \\ &= [n = 0] \qquad (\text{by Proposition 1.3.23 in Lecture 7}) \\ &= 0 \qquad (\text{since } n \text{ is positive, so that } n \text{ cannot be } 0) . \end{aligned}$$

These two equations can be solved for e and o (just like any solvable system of 2 linear equations in 2 unknowns); the result is

$$e = o = \frac{2^n}{2} = 2^{n-1}.$$

This proves the proposition. $\square$

The next property of BCs is highly useful:

Proposition 1.3.28 (trinomial revision formula)**.** Let $n, a, b \in \mathbb{R}$. Then,

$$\binom{n}{a}\binom{a}{b} = \binom{n}{b}\binom{n-b}{a-b}.$$

Proof. Four cases are possible:

Case 1: We have $b \notin \mathbb{N}$.

Case 2: We have $b \in \mathbb{N}$ and $a \notin \mathbb{N}$.

Case 3: We have $b \in \mathbb{N}$ and $a \in \mathbb{N}$ and $a < b$.

Case 4: We have $b \in \mathbb{N}$ and $a \in \mathbb{N}$ and $a \geq b$.

In each of the Cases 1, 2 and 3, the claim boils down to $0 = 0$ (check it![1]). So we only need to consider Case 4. In this case,

$$\begin{aligned}
\binom{n}{a} &= \frac{n(n-1)(n-2)\cdots(n-a+1)}{a!}; \\
\binom{a}{b} &= \frac{a!}{b!(a-b)!} \qquad \left(\begin{array}{c}\text{by the factorial formula,} \\ \text{since } b \in \mathbb{N} \text{ and } a \in \mathbb{N} \text{ and } a \geq b\end{array}\right); \\
\binom{n}{b} &= \frac{n(n-1)(n-2)\cdots(n-b+1)}{b!}; \\
\binom{n-b}{a-b} &= \frac{(n-b)(n-b-1)(n-b-2)\cdots(n-a+1)}{(a-b)!}.
\end{aligned}$$

So we only need to prove that

$$\begin{aligned}
&\frac{n(n-1)(n-2)\cdots(n-a+1)}{a!} \cdot \frac{a!}{b!(a-b)!} \\
&= \frac{n(n-1)(n-2)\cdots(n-b+1)}{b!} \cdot \frac{(n-b)(n-b-1)(n-b-2)\cdots(n-a+1)}{(a-b)!}.
\end{aligned}$$

Cancelling out all the factorials, we can simplify this to

$$\begin{aligned}
&n(n-1)(n-2)\cdots(n-a+1) \\
&= (n(n-1)(n-2)\cdots(n-b+1)) \cdot ((n-b)(n-b-1)(n-b-2)\cdots(n-a+1)).
\end{aligned}$$

But this is clear from basic properties of finite products, since $0 \leq b \leq a$. So the trinomial revision formula is proved. $\square$

[1] Details can be found in §2.2.1 of the 2019 notes.

As a particular case of trinomial revision, we obtain the **absorption formula**:

Proposition 1.3.29 (absorption formula)**.** Let $n, m \in \mathbb{R}$ such that $n \neq 0$. Then,

$$\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1}.$$

Proof. Apply the trinomial revision formula to m, n and 1 instead of n, a and b. Then, recall that $\binom{m}{1} = m$ and $\binom{n}{1} = n$. ☐

Here are some less trivial properties of BCs. Note that these are variations (and in some sense, generalizations) of the binomial theorem.

Theorem 1.3.30 (Chu–Vandermonde identity, aka Vandermonde convolution)**.** Let $n \in \mathbb{N}$, and let $x, y \in \mathbb{R}$. Then,

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}.$$

Theorem 1.3.31 (Cauchy's binomial formula)**.** Let $n \in \mathbb{N}$ and $x, y, z \in \mathbb{R}$. Then,

$$\sum_{k=0}^{n} \binom{n}{k} (x+kz)^k (y-kz)^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!} (x+y)^k z^{n-k}.$$

Theorem 1.3.32 (Abel's binomial formula)**.** Let $n \in \mathbb{N}$ and $x, y, z \in \mathbb{R}$. Then,

$$\sum_{k=0}^{n} \binom{n}{k} \underbrace{x (x+kz)^{k-1}}_{\substack{\text{This should be} \\ \text{read as 1} \\ \text{if } k=0}} (y-kz)^{n-k} = (x+y)^n .$$

We will prove the first of these three theorems in Lecture 15; the other two will not be proved in this course (but their proofs can be found in [18f-mt3s, Exercise 1][2]). Note that the original binomial formula can be derived from any of Cauchy's and Abel's binomial formulas by setting $z = 0$.

Identities that involve BCs (like the ones we have discussed above) are called **binomial identities**. There are many of them; I've seen whole books devoted to them. Here is another example of a binomial identity:

[2]More precisely, [18f-mt3s, Exercise 1] proves them in the case when $z = 1$. However, the general case can be easily obtained from this case (just replace x, y, z by $\frac{x}{z}, \frac{y}{z}, 1$).

Proposition 1.3.33. Let $n \in \mathbb{N}$. Then,

$$\sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1}.$$

First proof (sketched). The absorption formula yields $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ for each $k \neq 0$. Thus, $k\binom{n}{k} = n\binom{n-1}{k-1}$ for each $k \neq 0$. This also holds for $k = 0$ for pretty trivial reasons. Hence, if $n > 0$, then

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=0}^{n} n\binom{n-1}{k-1} = n \underbrace{\sum_{k=0}^{n} \binom{n-1}{k-1}}_{\substack{= \sum\limits_{k=-1}^{n-1} \binom{n-1}{k} = \sum\limits_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1} \\ \text{(by Corollary 1.3.22 in} \\ \text{Lecture 7, since } n>0\text{)}}} = n \cdot 2^{n-1}.$$

You can easily check that this also holds for $n = 0$. Thus, the proposition is proved. □

Second proof (sketched). The binomial formula (applied to $y = 1$) yields

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (x+1)^n \qquad \text{for all } x \in \mathbb{R}.$$

Take the derivative with respect to x. This results in

$$\sum_{k=0}^{n} \binom{n}{k} \cdot kx^{k-1} = n \cdot (x+1)^{n-1}.$$

Evaluating this at $x = 1$, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} \cdot k = n \cdot (1+1)^{n-1} = n \cdot 2^{n-1},$$

which is the claim of Proposition 1.3.33. □

We will see yet another proof of Proposition 1.3.33 eventually (Remark 1.6.5 in Lecture 11).

1.4. Counting subsets

1.4.1. All subsets

We have already seen that if S is an n-element set, and $k \in \mathbb{R}$, then

$$(\text{\# of } k\text{-element subsets of } S) = \binom{n}{k}.$$

So much for counting subsets of a given size. But what about counting **all** subsets? The answer is really simple:

Theorem 1.4.1. Let S be an n-element set (where $n \in \mathbb{N}$). Then,

$$(\text{\# of subsets of } S) = 2^n.$$

Proof. Analogous to the proof of

$$(\text{\# of } k\text{-element subsets of } S) = \binom{n}{k}.$$

This time, instead of Pascal's recurrence, we have to use the identity $2^{n-1} + 2^{n-1} = 2^n$. □

We will soon see some more artful proofs of the theorem.

1.4.2. Lacunar subsets

Let us move on to counting not all, but only some subsets, based on some conditions that we want them to satisfy. Here is a simple-sounding one:

Definition 1.4.2. A set S of integers is said to be **lacunar** if it contains no two consecutive integers (i.e., there is no integer i such that both i and $i+1$ belong to S).

For instance, the set $\{1, 5, 7\}$ is lacunar, whereas $\{1, 5, 6\}$ is not. Note that any 1-element or 0-element set is lacunar.

The lacunar subsets of $[5]$ are

$$\begin{array}{llllll} \varnothing, & \{1\}, & \{2\}, & \{3\}, & \{4\}, & \{5\}, \\ \{1,3\}, & \{1,4\}, & \{1,5\}, & \{2,4\}, & \{2,5\}, & \\ \{3,5\}, & \{1,3,5\}. & & & & \end{array}$$

The word "lacunar" comes from "lacuna" (= "gap"). The same concept is known under different names (essentially, every author invents their own terminology). You can think of a lacunar set as a set of integers that value their privacy.

We can now ask several natural questions:

Question 1.4.3. For given $n, k \in \mathbb{N}$:
(a) How many lacunar subsets does $[n]$ have?
(b) How many k-element lacunar subsets does $[n]$ have?
(c) What is the largest size of a lacunar subset of $[n]$?

Let us start with part **(c)**. The answer is easiest to state in terms of ceilings:

Definition 1.4.4. Let $x \in \mathbb{R}$. Then:

- We let $\lfloor x \rfloor$ denote the largest integer $\leq x$. This integer $\lfloor x \rfloor$ is called the **floor** of x, or "**rounding down**" x.
- We let $\lceil x \rceil$ denote the smallest integer $\geq x$. This integer $\lceil x \rceil$ is called the **ceiling** of x, or "**rounding up**" x.

For example:

$$\begin{aligned} \lfloor 3 \rfloor = \lceil 3 \rceil = 3; \\ \lfloor 3.7 \rfloor = 3, \qquad \lceil 3.7 \rceil = 4; \\ \left\lfloor \sqrt{2} \right\rfloor = 1, \qquad \left\lceil \sqrt{2} \right\rceil = 2; \\ \lfloor -2.5 \rfloor = -3, \qquad \lceil -2.5 \rceil = -2. \end{aligned}$$

Now we can state the answer to Question 1.4.3 **(c)**:

Proposition 1.4.5. Let $n \in \mathbb{N}$. Then, the largest size of a lacunar subset of $[n]$ is $\lceil n/2 \rceil$.

Our proof of this will rely on the following basic fact:

Theorem 1.4.6 (difference rule)**.** Let A be a finite set. Let B be a subset of A. Then:
(a) We have $|A \setminus B| = |A| - |B|$.
(b) We have $|B| \leq |A|$.
(c) If $|B| = |A|$, then $B = A$.

This theorem is basic set theory, so we won't prove it. Part **(a)** is known as the *difference rule*.

Proof of Proposition 1.4.5. The set

$$\{\text{all odd elements of } [n]\} = \{1 < 3 < 5 < \cdots < (n \text{ or } n-1)\}$$

(the $<$-signs on the right hand side here are simply saying that the elements are being listed in increasing order) is a lacunar subset of $[n]$ whose size is $\lceil n/2 \rceil$.

So the size $\lceil n/2 \rceil$ is reachable. It remains to show that this size is maximal, i.e., that every lacunar subset of $[n]$ has size $\leq \lceil n/2 \rceil$.

So let L be a lacunar subset of $[n]$. Let L^+ be its "shadow", defined by

$$L^+ := \{\ell + 1 \mid \ell \in L\}.$$

Note that L^+ is a subset of $\{2, 3, \ldots, n+1\}$. Therefore, both L and L^+ are subsets of $[n+1]$. Hence, $L \cup L^+ \subseteq [n+1]$. Theorem 1.4.6 **(b)** thus yields $|L \cup L^+| \leq |[n+1]| = n+1$.

However, the two sets L and L^+ are disjoint (because if they had an element k in common, then we would have $k \in L$ and $k - 1 \in L$ (since $k \in L^+$), and thus the two consecutive integers $k-1$ and k would both belong to L, which would contradict the lacunarity of L). Hence, by the sum rule, we obtain $|L \cup L^+| = |L| + |L^+|$. However, $|L^+| = |L|$ (by the bijection rule, because the map $L \to L^+$, $\ell \mapsto \ell + 1$ is a bijection). Thus,

$$\left|L \cup L^+\right| = |L| + \underbrace{\left|L^+\right|}_{=|L|} = |L| + |L| = 2 \cdot |L|,$$

so that

$$2 \cdot |L| = \left|L \cup L^+\right| \leq n + 1.$$

Hence, $|L| \leq \dfrac{n+1}{2}$. How can we derive $|L| \leq \lceil n/2 \rceil$ from this?

One way is to distinguish between the cases "n even" and "n odd", and make a (very simple) computation in each case. Another way is to argue that

$$|L| \leq \frac{n+1}{2} = \underbrace{\frac{n}{2}}_{\leq \left\lceil \frac{n}{2} \right\rceil} + \underbrace{\frac{1}{2}}_{<1} < \left\lceil \frac{n}{2} \right\rceil + 1,$$

and therefore (since both $|L|$ and $\left\lceil \dfrac{n}{2} \right\rceil + 1$ are integers) we get[3]

$$|L| \leq \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) - 1 = \left\lceil \frac{n}{2} \right\rceil.$$

So we have shown that the lacunar subset L has size $\leq \lceil n/2 \rceil$. This completes the proof of Proposition 1.4.5. □

We note in passing that Theorem 1.4.6 **(b)** also helps us prove the identity $\sum\limits_{k=0}^{n} \dbinom{n}{k} = 2^n$ (which we derived from the binomial formula in Lecture 7) combinatorially:

[3] Here, we are using the fact that if x and y are two integers satisfying $x < y$, then $x \leq y - 1$.

Combinatorial proof of $\sum\limits_{k=0}^{n} \dbinom{n}{k} = 2^n$. Let $n \in \mathbb{N}$. Then, each subset S of $[n]$ has size $|S| \in \{0, 1, \ldots, n\}$ (since Theorem 1.4.6 **(b)** yields $|S| \leq |[n]| = n$). Thus, by the sum rule, we have

$$\begin{aligned}
&(\text{\# of all subsets } S \text{ of } [n]) \\
&= \sum_{k=0}^{n} \underbrace{(\text{\# of all subsets } S \text{ of } [n] \text{ that have size } k)}_{=(\text{\# of all } k\text{-element subsets of } [n]) = \dbinom{n}{k}} \\
&= \sum_{k=0}^{n} \dbinom{n}{k}.
\end{aligned}$$

Hence, $\sum\limits_{k=0}^{n} \dbinom{n}{k} = (\text{\# of all subsets } S \text{ of } [n]) = 2^n$ (by Theorem 1.4.1), qed. $\square$

References

[18f-mt3s] Darij Grinberg, *UMN Fall 2018 Math 5705 midterm #3 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/18f/mt3s.pdf`