Math 222 Fall 2022, Lecture 5: Introduction

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

1. Introduction (cont'd)

1.2. Sums of powers

1.2.3. Some rules for sums (cont'd)

The last summation rule we saw last time says the following:

- **Splitting a sum by a value of a function:** Let S and W be two finite sets. Let $f : S \to W$ be a map. Let a_s be a number for each $s \in S$. Then,

$$\sum_{s \in S} a_s = \sum_{w \in W} \sum_{\substack{s \in S; \\ f(s) = w}} a_s \qquad (1)$$

  (the right hand side is a sum of sums, which is why you are seeing two $\sum$ signs side by side).

This splitting rule can be used to derive a slightly more general sum rule for combinatorics:

> **Theorem 1.2.4** (the sum rule, in summation-sign form)**.** Let S and W be two finite sets. Let $f : S \to W$ be a map. Then,
>
> $$|S| = \sum_{w \in W} (\text{\# of } s \in S \text{ satisfying } f(s) = w).$$

For example, if you have a finite set of socks, each of which is either red or green or blue, then

$$(\text{\# of socks}) = (\text{\# of red socks}) + (\text{\# of green socks}) + (\text{\# of blue socks}).$$

(This follows by applying Theorem 1.2.4 to $S = \{\text{socks}\}$, $W = \{\text{colors}\}$ and $f(s) = (\text{color of the sock } s)$.)

Proof of Theorem 1.2.4. Applying (1) to $a_s = 1$, we find

$$\sum_{s \in S} 1 = \sum_{w \in W} \sum_{\substack{s \in S; \\ f(s) = w}} 1.$$

However, in view of the equalities

$$\sum_{s\in S} 1 = |S| \cdot 1 = |S| \qquad \text{and}$$

$$\sum_{\substack{s\in S;\\ f(s)=w}} 1 = (\text{\# of } s \in S \text{ satisfying } f(s) = w) \cdot 1$$

$$= (\text{\# of } s \in S \text{ satisfying } f(s) = w),$$

this can be rewritten as

$$|S| = \sum_{w\in W} (\text{\# of } s \in S \text{ satisfying } f(s) = w),$$

and this is precisely the claim of Theorem 1.2.4. □

1.2.4. A few words about finite products

Finite products are defined in the same way as finite sums, but using the operation $\cdot$ instead of $+$.

The product of a family of numbers a_s is denoted by $\prod_{s\in S} a_s$.

Almost all rules for finite sums have analogues for finite products.[1]

The analogue of an empty sum $\sum_{s\in\varnothing} a_s$ is an empty product $\prod_{s\in\varnothing} a_s$. This empty product is defined to be 1. In particular, this entails that $x^0 = 1$ for any number x, because the m-th power x^m (for $m \in \mathbb{N}$) is defined to be the finite product $\underbrace{xx\cdots x}_{m \text{ many } x\text{'s}} = \prod_{i=1}^{m} x$.

1.2.5. The sums $1^k + 2^k + ... + n^k$

So we know that the sum of the first n positive integers is $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ for any $n \in \mathbb{N}$. What about summing higher powers (e.g., squares or cubes) of the first n positive integers?

[1]Why "almost"? Only because of one trivial exception: Finite sums satisfy the "generalized distributivity law" $\sum_{s\in S} \lambda a_s = \lambda \sum_{s\in S} a_s$ (for any number λ and any numbers a_s). The analogue of this law for products takes this form

$$\prod_{s\in S} a_s^{\lambda} = \left(\prod_{s\in S} a_s\right)^{\lambda},$$

but holds only for $\lambda \in \mathbb{N}$ (or for $\lambda \in \mathbb{Z}$ if we assume that all a_s are nonzero). If we try to apply it (say) to $\lambda = 1/2$, we get wrong results such as $(-1)^{1/2} (-1)^{1/2} = 1^{1/2}$.

Theorem 1.2.5. Let $n \in \mathbb{N}$. Then,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n\left(n+1\right)\left(2n+1\right)}{6}.$$

Theorem 1.2.6. Let $n \in \mathbb{N}$. Then,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2\left(n+1\right)^2}{4}.$$

These theorems can again be proved by induction or in various other ways (see §1.2.5 in the 2019 notes for references).

There are similar formulas for sums of 4th and 5th powers. Using the summation sign:

$$\begin{aligned}
\sum_{i=1}^{n} i^0 &= n; \\
\sum_{i=1}^{n} i^1 &= \frac{n\left(n+1\right)}{2}; \\
\sum_{i=1}^{n} i^2 &= \frac{n\left(n+1\right)\left(2n+1\right)}{6}; \\
\sum_{i=1}^{n} i^3 &= \frac{n^2\left(n+1\right)^2}{4}; \\
\sum_{i=1}^{n} i^4 &= \frac{n\left(2n+1\right)\left(n+1\right)\left(3n+3n^2-1\right)}{30}; \\
\sum_{i=1}^{n} i^5 &= \frac{n^2\left(n+1\right)^2\left(2n+2n^2-1\right)}{12}.
\end{aligned}$$

How does this continue (if it does)? It looks like for any given (constant) $k \in \mathbb{N}$, the sum $\sum_{i=1}^{n} i^k$ can be written as a degree-$(k+1)$ polynomial in n (as long as $n \in \mathbb{N}$). Why, and how do we find this polynomial?

Here is an answer, which we will prove in the next chapter:

Theorem 1.2.7. Let $n \in \mathbb{N}$, and let k be a positive integer. Then,

$$\sum_{i=1}^{n} i^k = \sum_{i=0}^{k} \operatorname{sur}\left(k, i\right) \cdot \binom{n+1}{i+1},$$

where

- sur (k,i) denotes the # of surjective (= onto) maps from $[k]$ to $[i]$;
- $\dbinom{n+1}{i+1}$ is an instance of a binomial coefficient. The binomial coefficient $\dbinom{m}{k}$ is defined by

$$\binom{m}{k} = \frac{m\,(m-1)\,(m-2)\cdots(m-k+1)}{k\,(k-1)\,(k-2)\cdots 1}$$

for any number m and any $k \in \mathbb{N}$ (we will soon extend this definition to all k); for fixed k, this is a degree-k polynomial in m.

This theorem is actually useful in computing $\sum\limits_{i=1}^{n} i^k$: Even if the right hand side is given as a finite sum, this sum has only $k+1$ addends, so it does not get more complicated with increasing n.

For example, we can rederive our above formula for $\sum\limits_{i=1}^{n} i^2$ from Theorem 1.2.7: Namely, plugging $k=2$ into Theorem 1.2.7, we get

$$\begin{aligned}
&\sum_{i=1}^{n} i^2 \\
&= \sum_{i=0}^{2} \operatorname{sur}(2,i)\cdot\binom{n+1}{i+1} \\
&= \underbrace{\operatorname{sur}(2,0)}_{\substack{=0 \\ \text{(since there are no} \\ \text{surjective maps } [2]\to[0], \\ \text{or any maps } [2]\to[0] \\ \text{for that matter)}}} \cdot \binom{n+1}{0+1} + \underbrace{\operatorname{sur}(2,1)}_{\substack{=1 \\ \text{(since there is exactly} \\ \text{one surjective} \\ \text{map } [2]\to[1])}} \cdot \binom{n+1}{1+1} + \underbrace{\operatorname{sur}(2,2)}_{\substack{=2 \\ \text{(since there are} \\ \text{two surjective} \\ \text{maps } [2]\to[2])}} \cdot \binom{n+1}{2+1} \\
&= 0\cdot\binom{n+1}{0+1} + 1\cdot\binom{n+1}{1+1} + 2\cdot\binom{n+1}{2+1} \\
&= \binom{n+1}{1+1} + 2\cdot\binom{n+1}{2+1} = \binom{n+1}{2} + 2\cdot\binom{n+1}{3} \\
&= \frac{(n+1)\,n}{2\cdot 1} + 2\cdot\frac{(n+1)\,n\,(n-1)}{3\cdot 2\cdot 1} = \frac{n\,(n+1)\,(2n+1)}{6}.
\end{aligned}$$

So we recover our formula for the sum of the first n squares (Theorem 1.2.5).

We won't prove Theorem 1.2.7 right now, but we will take a closer look at the expressions that appear in it: first, the factorials and the binomial coefficients.

1.3. Factorials and binomial coefficients

1.3.1. Factorials

Definition 1.3.1. For any $n \in \mathbb{N}$, we define a positive integer $n!$ by

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1 = \prod_{i=1}^{n} i.$$

This integer $n!$ is called "n **factorial**".

So we have

$$\begin{aligned} 0! &= 1 \cdot 2 \cdot \cdots \cdot 0 = \prod_{i=1}^{0} i = (\text{empty product}) = 1; \\ 1! &= 1 = 1; \\ 2! &= 1 \cdot 2 = 2; \\ 3! &= 1 \cdot 2 \cdot 3 = 6; \\ 4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24; \\ 5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120; \\ 6! &= 720; \\ 7! &= 5\,040; \\ 8! &= 40\,320. \end{aligned}$$

(Yes, these numbers grow fast! See Stirling's approximation for about how fast they grow.)

Factorials can be computed recursively:

Proposition 1.3.2. If n is a positive integer, then $n! = (n-1)! \cdot n$.

Proof. Just split off the factor n from $n! = 1 \cdot 2 \cdot \cdots \cdot n$. The remaining factors form a product that equals $(n-1)!$. □

Factorials can be used to describe quite a few other things; the following fun exercise (Exercise 1.3.1 in the 2019 notes) is a simple example:

Exercise 1. Let $n \in \mathbb{N}$. Prove that the product of the first n **odd** positive integers is

$$1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) = \frac{(2n)!}{2^n n!}.$$

1.3.2. Definition of the binomial coefficients

We can now define binomial coefficients in the generality I want to define them in:

Definition 1.3.3. Let n and k be any two numbers (i.e., real or complex numbers). We define a number $\binom{n}{k}$ as follows:

- If $k \in \mathbb{N}$ (recall that $0 \in \mathbb{N}$), then
$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$
- If $k \notin \mathbb{N}$, then
$$\binom{n}{k} = 0.$$

This number $\binom{n}{k}$ is pronounced "n **choose** k" and is called a **binomial coefficient**.

This definition is standard across much of the literature, but some authors nevertheless disagree with it, mostly in situations when n is negative or k is a non-integer. So be careful with some texts and papers.

Some authors use the notations C_k^n or ${}^nC_k$ or ${}_nC_k$ for $\binom{n}{k}$.

Do not mistake the binomial coefficient $\binom{n}{k}$ for the size-2 vector $\begin{pmatrix} n \\ k \end{pmatrix}$. They look very similar (the only real difference is the spacing); fortunately they rarely appear in the same context.

The above definition of $\binom{n}{k}$ is worth memorizing. The product
$$n(n-1)(n-2)\cdots(n-k+1)$$
in the numerator is a product of k factors, the first of which is n while each following factor equals the previous factor minus 1. Let's apply the definition to compute some examples:

Example 1.3.4. **(a)** For any number n, we have
$$\binom{n}{0} = \frac{(\text{empty product})}{0!} = \frac{1}{1} = 1.$$

(b) For any number n, we have
$$\binom{n}{1} = \frac{n}{1!} = \frac{n}{1} = n.$$

(c) For any number n, we have

$$\binom{n}{2} = \frac{n(n-1)}{2!} = \frac{n(n-1)}{2}.$$

(d) For any number n, we likewise have

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

(e) For any $k \in \mathbb{N}$, we have

$$\begin{aligned}\binom{-1}{k} &= \frac{(-1)(-1-1)(-1-2)\cdots(-1-k+1)}{k!} \\ &= \frac{(-1)(-2)(-3)\cdots(-k)}{k!} = \frac{(-1)^k \cdot 1 \cdot 2 \cdot \cdots \cdot k}{k!} = \frac{(-1)^k \cdot k!}{k!} = (-1)^k.\end{aligned}$$

(f) We have

$$\binom{2}{\sqrt{2}} = 0 \qquad \left(\text{since } \sqrt{2} \notin \mathbb{N}\right)$$

but

$$\binom{\sqrt{2}}{2} = \frac{\sqrt{2}\left(\sqrt{2}-1\right)}{2} \neq 0.$$

It would be nice to have the binomial coefficients all nicely tabulated; alas, there are infinitely many of them, so the table would take a bit too much space. Here is a table of the binomial coefficients $\binom{n}{k}$ for all $n \in \{-3, -2, -1, \ldots, 6\}$ and some of the $k \in \{0, 1, 2, 3, 4, 5\}$ (this is a somewhat unusual table, since the values of n correspond to the rows, but the values of k correspond to **diagonals**, not columns):

											$k{=}0$ $\swarrow$		$k{=}1$ $\swarrow$		$k{=}2$ $\swarrow$		$k{=}3$ $\swarrow$
$n = -3$ $\to$										1		-3		6		-10	
$n = -2$ $\to$									1		-2		3		-4		
$n = -1$ $\to$								1		-1		1		-1		1	
$n = 0$ $\to$							1		0		0		0		0		
$n = 1$ $\to$						1		1		0		0		0		0	
$n = 2$ $\to$					1		2		1		0		0		0		
$n = 3$ $\to$				1		3		3		1		0		0		0	
$n = 4$ $\to$			1		4		6		4		1		0		0		
$n = 5$ $\to$		1		5		10		10		5		1		0		0	
$n = 6$ $\to$	1		6		15		20		15		6		1		0		

1.3.3. Fundamental properties of the binomial coefficients

By looking closely at this table, we can spot several important properties of binomial coefficients. Here, for one, is the explanation for the many 0's in the right half of the table:

Proposition 1.3.5. If $n \in \mathbb{N}$ and $k \in \mathbb{R}$ satisfy $k > n$, then $\dbinom{n}{k} = 0$.

Proof. Let $n \in \mathbb{N}$ and $k \in \mathbb{R}$ satisfy $k > n$. If $k \notin \mathbb{N}$, then $\dbinom{n}{k} = 0$ by definition. So assume WLOG that $k \in \mathbb{N}$. Then, the definition of $\dbinom{n}{k}$ yields

$$\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k!}.$$

However, $k > n$ and $k \in \mathbb{N}$ entail that one of the factors $n, n-1, n-2, \ldots, n-k+1$ in the numerator is 0. So the whole product is 0, and with it $\dbinom{n}{k}$. $\square$

Warning: Proposition 1.3.5 does not hold if we drop the $n \in \mathbb{N}$ requirement. For instance, $\dbinom{-1}{2} \neq 0$ even though $2 > -1$.