Math 222 Fall 2022, Lecture 3: Introduction

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

1. Introduction (cont'd)

1.1. Domino tilings (cont'd)

1.1.5. The $m = 2$ case and the Fibonacci sequence (cont'd)

Recall how we defined the Fibonacci sequence last time:

Definition 1.1.9. The **Fibonacci sequence** is the sequence $(f_0, f_1, f_2, \ldots)$ of nonnegative integers defined recursively by

$$f_0 = 0, \qquad f_1 = 1, \qquad \text{and} \qquad f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2.$$

Here are the first few Fibonacci numbers (= entries of the Fibonacci sequence):

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
f_n	0	1	1	2	3	5	8	13	21	34	$\cdots$

.

Last time we proved the following:

Proposition 1.1.10. We have $d_{n,2} = f_{n+1}$ for each $n \in \mathbb{N}$.

Is this an answer to the counting question for $R_{n,2}$? How easy is it to compute a given Fibonacci number f_n ?

The recursive definition is good but maybe something better exists? Yes, as it turns out:

Theorem 1.1.11 (Binet's formula)**.** For each $n \in \mathbb{N}$, we have

$$f_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \psi^n\right),$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots \qquad \text{and} \qquad \psi = \frac{1-\sqrt{5}}{2} \approx -0.618\ldots.$$

A few remarks are in order:

- The numbers φ and ψ are the two roots of the quadratic polynomial $x^2 - x - 1$. In other words, they are the solutions of the equation $x^2 = x + 1$.
- Binet's formula can be used to compute f_n, but you have to be careful. Numerically computing $\varphi^n - \psi^n$ will often (for $n > 100$ or so) be marred by progressively worse approximation errors, and you will get a wrong result for f_n even if you round to the nearest integer. The correct way to use Binet's formula is to work algebraically – i.e., perform exact computations with numbers of the form $a + b\sqrt{5}$ where $a, b \in \mathbb{Q}$. See the 2019 notes for a bit more detail on how this works.
- Yes, it's a strange thing: f_n is a nonnegative integer, but the explicit formula involves irrationalities. But it's not the only time that something like this happens in mathematics.

Proof of Theorem 1.1.11. There is a straightforward proof by strong induction (same idea as in the last proof we gave): We must show that the two sequences

$$(f_0, f_1, f_2, \ldots) \qquad \text{and} \qquad \left(\frac{1}{\sqrt{5}}\left(\varphi^0 - \psi^0\right),\ \frac{1}{\sqrt{5}}\left(\varphi^1 - \psi^1\right),\ \frac{1}{\sqrt{5}}\left(\varphi^2 - \psi^2\right),\ \ldots\right)$$

are identical. To do so, it suffices to check that

- the first two entries of both sequences are the same;
- both sequences satisfy the same recursive rule, viz., that each entry (starting with the third entry) equals the sum of the preceding two entries.

This is just a matter of computation. The second part boils down to showing that

$$\frac{1}{\sqrt{5}}\left(\varphi^n - \psi^n\right) = \frac{1}{\sqrt{5}}\left(\varphi^{n-1} - \psi^{n-1}\right) + \frac{1}{\sqrt{5}}\left(\varphi^{n-2} - \psi^{n-2}\right)$$

for each $n \geq 2$; but this follows from

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2} \qquad \text{and} \qquad \psi^n = \psi^{n-1} + \psi^{n-2}.$$

□

So Binet's formula is proved. The question of how to find such a formula is more interesting, but we won't discuss it for now. The theory behind this is the **theory of linear recurrences**.

With Binet's formula, we have an explicit expression for $d_{n,2}$, so our counting problem is solved for the $n \times 2$-rectangle $R_{n,2}$.

1.1.6. Kasteleyn's formula (teaser)

What about larger rectangles? Here are two recursive formulas for domino tilings of $R_{n,3}$ and $R_{n,4}$:

Proposition 1.1.12. We have

$$d_{n,3} = 4d_{n-2,3} - d_{n-4,3} \qquad \text{for each } n \geq 4.$$

Proposition 1.1.13. We have

$$d_{n,4} = d_{n-1,4} + 5d_{n-2,4} + d_{n-3,4} - d_{n-4,4} \qquad \text{for each } n \geq 4.$$

These are harder to prove (and also messier) than our $d_{n,2}$ recurrence; we will not show the proofs here. As you may have guessed, there is a pattern here: For any fixed $m \in \mathbb{N}$, the sequence $(d_{0,m}, d_{1,m}, d_{2,m}, \ldots)$ follows a linear recursion (i.e., for any n sufficiently high, $d_{n,m}$ can be written as a constant-term linear combination of $d_{k,m}$'s with $k < n$), but the recursion gets more complicated the larger m is. As it comes to computing $d_{n,m}$ in general, this is likely a dead end.

This does not mean, however, that the domino tiling counting problem is hopeless! In 1961, P. W. Kasteleyn (a physicist) found an explicit (in an appropriate sense) formula for $d_{n,m}$ that holds in full generality:

Theorem 1.1.14 (Kasteleyn's formula)**.** Assume that m is even and $n \geq 1$. Then,

$$d_{n,m} = 2^{mn/2} \prod_{j=1}^{m/2} \prod_{k=1}^{n} \sqrt{\left(\cos \frac{j\pi}{m+1}\right)^2 + \left(\cos \frac{k\pi}{n+1}\right)^2}.$$

(Here, we are using the product sign $\prod$: That is, if $a_1, a_2, \ldots, a_p$ are any numbers, then $\prod_{i=1}^{p} a_i$ means the product $a_1 a_2 \cdots a_p$. The presence of two product signs directly following one another means that we are taking a product of products.)

A few questions suggest themselves here:

- Why do cosines and π appear in a formula for a nonnegative integer?
- Why was a physicist studying domino tilings of a rectangle?
- How could one prove such a formula?

I will only spend a few words on either question, as this is graduate course material (perhaps even best left to a topics course):

- **Why was a physicist studying domino tilings:** A domino tiling is a (rather idealized) model for a liquid consisting of "dimers" (polymers that take up two adjacent sites in a rectangular lattice; these are exactly our dominos). Kasteleyn was studying the adsorption of molecules on a surface, so the lattice is indeed 2-dimensional and can be modeled by our rectangle $R_{n,m}$.

- **How is Theorem 1.1.14 proved:** Using some advanced linear algebra, specifically the notion of the *Pfaffian* of a skew-symmetric matrix (a variant of the determinant).

- **Why do cosines and π appear in the formula:** Because the matrix in question is a Kronecker product of two tridiagonal matrices, and the latter matrices are known to have eigenvalues $\left(\cos\frac{j\pi}{k+1}\right)^2$. The determinant of a matrix can be computed as the product of its eigenvalues, and the Pfaffian of a skew-symmetric matrix can be computed as the square root of its determinant (at least if it is ≥ 0).

We won't even get close to the actual proof of Theorem 1.1.14, but it is outlined or even exposed in full in some texts. Two references are given in my 2019 notes (Subsection 1.1.6).

Theorem 1.1.14 might appear like a pie-in-the-sky, but it can actually be used to compute $d_{n,m}$. For instance, it yields

$$d_{8,8} = 12\,988\,816.$$

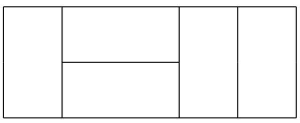
This is the answer to the question "how many ways are there to tile a chessboard with dominos?". Good luck finding all these tilings!

1.1.7. Axisymmetric domino tilings

Here is a variant of the problem of counting the domino tilings of $R_{n,2}$:

Exercise 1. Let $n \in \mathbb{N}$. Say that a domino tiling T of $R_{n,2}$ is **axisymmetric** if reflecting it across the vertical axis of symmetry of $R_{n,2}$ leaves it unchanged.

For example, the tilings



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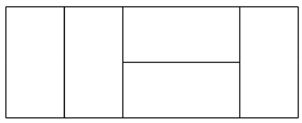
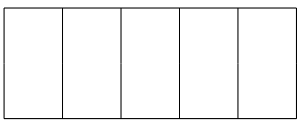


are not axisymmetric (indeed, reflecting them across the vertical line transforms them into one another, and they are not the same), but the tilings



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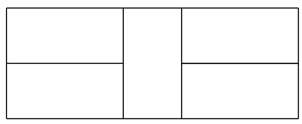
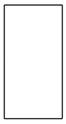
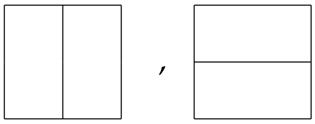
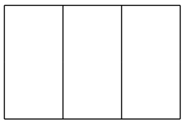


are axisymmetric.

For instance, let us list the axisymmetric domino tilings for $R_{n,2}$ when n is small:

n	axisymmetric domino tilings	their number
0		1
1		1
2		2
3		1
4		3

How many axisymmetric domino tilings T of $R_{n,2}$ are there?

Answer:

$$\begin{cases} f_{n/2+2}, & \text{if } n \text{ is even;} \\ f_{(n+1)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

See the 2019 notes (Subsection 1.1.7) for how to prove this. The main idea is to distinguish the cases n even and n odd, and consider the "left half" of the rectangle.

1.1.8. Tiling rectangles with k-bricks

An obvious generalization of dominos are **k-bricks**:

Fix a positive integer k for the rest of this section. Define a **k-brick** to be a set of k squares stacked upon each other either vertically or horizontally. In other words, it is a set of the form

$$\{(i,j)\,,\ (i+1,j)\,,\ \ldots,\ (i+k-1,j)\} = \underbrace{\boxed{}}_{\text{horizontal } k\text{-brick}} \qquad \text{or}$$

$$\{(i,j)\,,\ (i,j+1)\,,\ \ldots,\ (i,j+k-1)\} = \underbrace{\boxed{\rule{0pt}{6em}}}_{\text{vertical } k\text{-brick}} .$$

Which rectangles $R_{n,m}$ can be tiled with k-bricks? Here is an answer:

Proposition 1.1.15. Let $n, m \in \mathbb{N}$ and let k be a positive integer. Then, the rectangle $R_{n,m}$ has a k-brick tiling if and only if we have $k \mid m$ or $k \mid n$.

Proof. See the 2019 notes (Subsection 1.1.8). $\square$