Math 222 Fall 2022, Lecture 2: Introduction

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

1. Introduction (cont'd)

1.1. Domino tilings (cont'd)

1.1.2. The odd-by-odd case and the sum rule

Let us first handle a particularly simple case of our counting problem:

Proposition 1.1.1. Assume that n and m are both odd. Then, $d_{n,m} = 0$.

Proof. The total number of squares in $R_{n,m}$ is nm, which is odd (since n and m are odd). However, each domino has an even number of squares (namely, 2). So any shape that can be tiled by dominos must also have an even number of squares (since a sum of even numbers is even). But $R_{n,m}$ does not. So $R_{n,m}$ cannot be tiled by dominos. In other words, the # of ways to tile it is 0. But this is saying precisely that $d_{n,m} = 0$. □

This proof rests upon two fundamental principles, which are worth stating at least once.

The first one is the **sum rule**:

Theorem 1.1.2 (sum rule). If a finite set S is the union of k **disjoint** sets $S_1, S_2, \ldots, S_k$, then

$$|S| = |S_1| + |S_2| + \cdots + |S_k|.$$

In other words, for any k **disjoint** finite sets $S_1, S_2, \ldots, S_k$, we have

$$|S_1 \cup S_2 \cup \cdots \cup S_k| = |S_1| + |S_2| + \cdots + |S_k|.$$

The other principle we used is the following:

Theorem 1.1.3 (product rule for 2 sets). If X and Y are two finite sets, then $X \times Y$ is a finite set with size

$$|X \times Y| = |X| \cdot |Y|.$$

So here is our above proof of Proposition 1.1.1, formalized:

Proof of Proposition 1.1.1 (formal version). Let S be a domino tiling of $R_{n,m}$, where n and m are odd. Let $S_1, S_2, \ldots, S_k$ be the distinct dominos in S. Since the dominos in a domino tiling are disjoint, we then have

$$|S_1 \cup S_2 \cup \cdots \cup S_k| = |S_1| + |S_2| + \cdots + |S_k| \qquad \text{(by the sum rule)}.$$

However, since the dominos in a domino tiling of $R_{n,m}$ cover all of $R_{n,m}$, we have $S_1 \cup S_2 \cup \cdots \cup S_k = R_{n,m}$. So the previous equality can be rewritten as

$$|R_{n,m}| = \underbrace{|S_1|}_{=2} + \underbrace{|S_2|}_{=2} + \cdots + \underbrace{|S_k|}_{=2} = 2k,$$

which is even. On the other hand, from[1] $R_{n,m} = [n] \times [m]$, we obtain

$$|R_{n,m}| = |[n] \times [m]| = \underbrace{|[n]|}_{=n} \cdot \underbrace{|[m]|}_{=m} \qquad \text{(by the product rule)}$$
$$= nm,$$

which is odd (since n and m are both odd). Comparing these two equalities, we see that $|R_{n,m}|$ is both even and odd, which is impossible.

So we have found a contradiction for each domino tiling of $R_{n,m}$. This shows that there are no domino tilings of $R_{n,m}$. In other words, $d_{n,m} = 0$. Thus, Proposition 1.1.1 is proved again. □

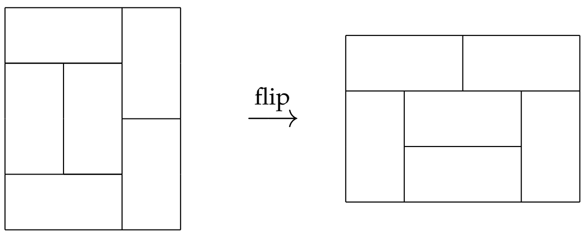
Usually, we won't formalize proofs like this, but I have deliberately done this here, so you know which principles underlie the simple intuitive argument we gave above. Any valid proof in combinatorics, no matter how colloquially it is explained, can be formalized into a rigorous argument based on set-theoretical foundations.

1.1.3. The symmetry and the bijection rule

Next, let us show a symmetry property of our tiling numbers $d_{n,m}$:

Proposition 1.1.4. Let $n, m \in \mathbb{N}$. Then, $d_{n,m} = d_{m,n}$.

Proof. The idea is very simple: The rectangle $R_{m,n}$ is just the rectangle $R_{n,m}$, flipped across the diagonal (i.e., the line with equation $x = y$). Thus, any domino tiling of $R_{n,m}$ can be likewise flipped, and this yields a domino tiling of $R_{m,n}$; and vice versa. Here is an example:



This shows that the domino tilings of $R_{n,m}$ are in one-to-one correspondence with the domino tilings of $R_{m,n}$. Hence, the # of the former equals the # of the latter. In other words, $d_{n,m} = d_{m,n}$. □

[1] Recall that $[k]$ means the set $\{1, 2, \ldots, k\}$.

Again, let me spell out the formal details of this argument (at least some of them) to exhibit another fundamental principle of combinatorics that underlies it. The principle says that if there is a one-to-one correspondence between the elements of some set and the elements of another set, then these two sets have the same size. More formally:

> **Theorem 1.1.5** (bijection principle)**.** If X and Y are two sets, and if $f : X \to Y$ is a bijective map (i.e., a bijection, i.e., a one-to-one correspondence), then
>
> $$|X| = |Y|.$$

Let me recall the definitions of in-/sur-/bijectivity, since these concepts are commonly known under other names:

- A map $f : X \to Y$ is said to be **injective** if it sends distinct elements to distinct elements (i.e., if it has the property that $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$). This is also known as a **one-to-one map**.
- A map $f : X \to Y$ is said to be **surjective** if each element of Y is a value of f (i.e., if it has the property that for each $y \in Y$, there exists an $x \in X$ such that $f(x) = y$). This is also known as an **onto map**.
- A map $f : X \to Y$ is said to be **bijective** if it is both injective and surjective. This is also known as a **one-to-one correspondence**. It is important to keep in mind that a map is bijective if and only if it is invertible (i.e., has an inverse map).

So let me sketch how the above proof of Proposition 1.1.4 can be made formal:

Proof of Proposition 1.1.4 (formal version). We want to apply the bijection principle to

$$X = \{\text{domino tilings of } R_{n,m}\} \qquad \text{and}$$
$$Y = \{\text{domino tilings of } R_{m,n}\}.$$

To do so, we need to construct a bijection from X to Y. The idea of this bijection is, of course, flipping the domino tiling across the $x = y$ diagonal. What does this mean formally? Well, a domino tilings consists of dominos, and a domino consists of squares, so let us first see how flipping works on squares.

To flip a square (i, j) across the diagonal simply means to replace it by (j, i). Thus, what we mean by "flipping squares" is a bijective map

$$F : R_{n,m} \to R_{m,n},$$
$$(i, j) \mapsto (j, i).$$

It is pretty clear that this map is bijective (its inverse map sends each (j,i) to (i,j)).

Next, let us define what it means to flip a domino: This means flipping both squares in the domino. That is, "flipping dominos" is a bijective map

$$\begin{aligned} F_{\text{dom}} : \{\text{dominos inside } R_{n,m}\} &\to \{\text{dominos inside } R_{m,n}\}, \\ D &\mapsto \{F(s) \mid s \in D\}. \end{aligned}$$

Finally, let us define flipping domino tilings: This is a bijective map

$$\begin{aligned} F_{\text{til}} : \{\text{domino tilings of } R_{n,m}\} &\to \{\text{domino tilings of } R_{m,n}\}, \\ T &\mapsto \{F_{\text{dom}}(D) \mid D \in T\}. \end{aligned}$$

Hence, the bijection principle yields

$$|\{\text{domino tilings of } R_{n,m}\}| = |\{\text{domino tilings of } R_{m,n}\}|.$$

In other words,

$$(\text{\# of domino tilings of } R_{n,m}) = (\text{\# of domino tilings of } R_{m,n}).$$

In other words, $d_{n,m} = d_{m,n}$. Thus, Proposition 1.1.4 is proved rigorously. □

One further note about the bijection principle: It has a converse:

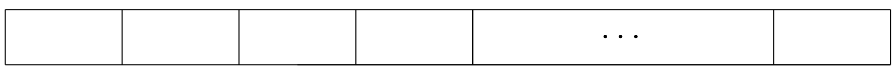
Theorem 1.1.6 (converse bijection principle)**.** If X and Y are two sets of the same size (that is, $|X| = |Y|$), then there exists a bijection from X to Y.

1.1.4. The $m = 1$ case

Next, let us solve a very simple particular case of our counting problem:

Proposition 1.1.7. Assume that $m = 1$ and n is even. Then, $d_{n,m} = 1$.

Proof. There is only one domino tiling of $R_{n,m} = R_{n,1}$, and it looks like this:



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The proof of this is a straightforward induction (argue that the first two columns must be covered by a horizontal domino, then argue that the next two columns must be covered by a horizontal domino, and so on, until all columns are accounted for). See the 2019 notes (proof of Proposition 1.1.8 in them, to be specific) for details. □

1.1.5. The $m = 2$ case and the Fibonacci sequence

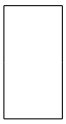
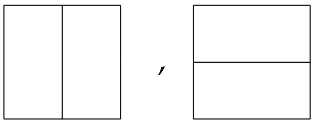
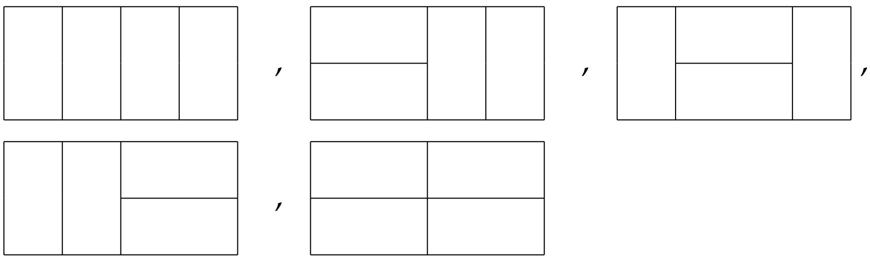
We have now fully covered the $m = 1$ case of our counting problem. But what about $m = 2$?

Here is a table of the $d_{n,2}$ values for $n \in \{0, 1, \ldots, 9\}$:

n	0	1	2	3	4	5	6	7	8	9
$d_{n,2}$	1	1	2	3	5	8	13	21	34	55

.

As an illustration, here are all domino tilings of $R_{n,2}$ for $n \in \{0, 1, 2, 3, 4\}$ (note that $R_{0,2}$ is the empty set and thus can be tiled by the empty set, which is a perfectly valid domino tiling):

n	$d_{n,m}$	domino tilings
0	$d_{0,2} = 1$	
1	$d_{1,2} = 1$	
2	$d_{2,2} = 2$	
3	$d_{3,2} = 3$	
4	$d_{4,2} = 5$	

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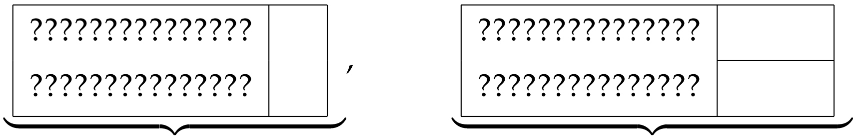
Is there any quick way of computing these numbers $d_{n,2}$, without having to find all domino tilings?

There is a **recursion** (i.e., a formula for $d_{n,2}$ in terms of previous $d_{k,2}$ values):

Proposition 1.1.8. For each integer $n \geq 2$, we have $d_{n,2} = d_{n-1,2} + d_{n-2,2}$.

Proof. Let $n \geq 2$ be an integer. Consider the last (i.e., rightmost) column of $R_{n,2}$ (formally speaking, this is the set $\{(n, 1), (n, 2)\}$).

In any domino tiling T of $R_{n,2}$, this last column is covered either by a vertical domino or by (parts of) two horizontal dominos. We call T a **type-1 tiling** in the former case and a **type-2 tiling** in the latter case. Visually, these look as follows:



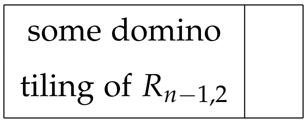
type-1 tiling, type-2 tiling

(where the question marks mean an unknown arrangement of dominos).

Note that any domino tiling of $R_{n,2}$ is either type-1 or type-2, but not both at the same time. Thus, by the sum rule, we have

$$(\text{\# of domino tilings of } R_{n,2}) = (\text{\# of type-1 tilings}) + (\text{\# of type-2 tilings}).$$

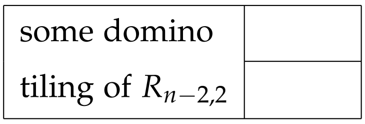
So let us compute the two addends on the RHS (= right-hand side).

A type-1 tiling consists of a vertical domino that covers the last column, and of a bunch of other dominos that cover the rest of the rectangle. But this rest is just the rectangle $R_{n-1,2}$. So a type-1 tiling is simply a domino tiling of $R_{n-1,2}$ with an extra vertical domino added at the rightmost end (visually, this looks as follows: 

). Thus,

$$\begin{aligned}(\text{\# of type-1 tilings}) &= (\text{\# of domino tilings of } R_{n-1,2}) \qquad (\text{by the bijection principle})\\ &= d_{n-1,2}\end{aligned}$$

(since $d_{n-1,2}$ was defined to be the # of domino tilings of $R_{n-1,2}$).

A type-2 tiling consists of two horizontal dominos that cover the last two columns, and of a bunch of other dominos that cover the rest of the rectangle. But this rest is just the rectangle $R_{n-2,2}$. So a type-2 tiling is simply a domino tiling of $R_{n-2,2}$ with two extra horizontal dominos added at the rightmost end (visually, this looks as follows: 

). Thus,

$$\begin{aligned}(\text{\# of type-2 tilings}) &= (\text{\# of domino tilings of } R_{n-2,2}) \qquad (\text{by the bijection principle})\\ &= d_{n-2,2}.\end{aligned}$$

Now, by the definition of $d_{n,2}$, we have

$$\begin{aligned}d_{n,2} &= (\text{\# of domino tilings of } R_{n,2}) = \underbrace{(\text{\# of type-1 tilings})}_{=d_{n-1,2}} + \underbrace{(\text{\# of type-2 tilings})}_{=d_{n-2,2}}\\ &= d_{n-1,2} + d_{n-2,2}, \qquad \text{qed.}\end{aligned}$$

(See the 2019 notes – specifically, the proof of Proposition 1.1.9 therein – for a more detailed version of this proof.) □

Proposition 1.1.8 provides a fairly efficient way to compute $d_{n,2}$ when n is not too large. But there are even better ways. First, let us recall a rather famous number sequence:

Definition 1.1.9. The **Fibonacci sequence** is the sequence $(f_0, f_1, f_2, \ldots)$ of nonnegative integers defined recursively by

$$f_0 = 0, \qquad f_1 = 1, \qquad \text{and} \qquad f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2.$$

This is a **recursive definition** – i.e., it tells us how to compute f_n given the previous entries of the sequence. For instance, in order to compute f_5, you first need to compute f_0, f_1, f_2, f_3, f_4 in this order.

When an object in mathematics is defined recursively, you can always wonder if there is also a direct way to compute it, without computing all the previous values. In particular, we can ask this question about the Fibonacci sequence. In the next lecture, we will see an answer!

Here is a table of the first few entries of the Fibonacci sequence:

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
f_n	0	1	1	2	3	5	8	13	21	34	$\cdots$

.

The entries f_n of this sequence are known as the **Fibonacci numbers**, and are one of the most popular concepts in elementary mathematics (see their Wikipedia page for some of their claims to fame). There are books, conferences and at least one journal devoted to them.

The recursive definition of the Fibonacci sequence (with its $f_n = f_{n-1} + f_{n-2}$ equation) is very similar to the recursive formula $d_{n,2} = d_{n-1,2} + d_{n-2,2}$ for our tiling numbers (Proposition 1.1.8). This shows that the sequence $(d_{0,2}, d_{1,2}, d_{2,2}, d_{3,2}, \ldots)$ and the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ have a lot in common: In both sequences, every entry starting with the third one is the sum of the two entries preceding it! However, the two sequences differ in their first entries (the former sequence starts with $d_{0,2} = 1$, while the latter starts with $f_0 = 0$). So the two sequences are not literally identical.

In truth, they are almost the same: The sequence $(d_{0,2}, d_{1,2}, d_{2,2}, d_{3,2}, \ldots)$ is just the Fibonacci sequence with its first entry removed! In other words:

Proposition 1.1.10. We have $d_{n,2} = f_{n+1}$ for each $n \in \mathbb{N}$.

Proof. We need to show that the sequences $(d_{0,2}, d_{1,2}, d_{2,2}, \ldots)$ and $(f_1, f_2, f_3, \ldots)$ are the same. To do, it suffices to notice that

- these sequences start with the same two values: $d_{0,2} = 1 = f_1$ and $d_{1,2} = 1 = f_2$.

- each further entry of either sequence is given by the same rule (namely, as the sum of the previous two entries). (This follows from Proposition 1.1.8.)

So the two sequences are equal.

(Formally speaking, this is an argument by strong induction; see the proof of Proposition 1.1.11 in the 2019 notes for the details.) □