Math 222 Fall 2022, Lecture 1: Introduction

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22fco`

0.1. Plan

My name is Darij Grinberg.

This is a course on enumerative combinatorics: the mathematics of finite sets and their sizes and the maps between them. It also includes the study of finite sums and binomial coefficients.

I will be using the lecture notes from the 2019 iteration of this course ("*Enumerative Combinatorics: class notes*", available at `https://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf`). I will refer to them as the "2019 notes", although they will be updated on occasion. I won't slavishly follow them; instead I will omit some details and sections while adding some new sections.

Everything I'm typing in class will also go on the course website (but it won't be as polished and fleshed-out as the 2019 notes). The website also serves as a syllabus, containing homework sets, grading policies, links to relevant systems (Gradescope, Piazza and Blackboard), and literature suggestions for further study. **Read the website!**

The course will be split into 5 chapters, whose content I shall briefly survey now:

1. **Introduction.** Here we will meet some counting problems and even solve a few of them. We'll introduce factorial and binomial coefficients and prove some basic identities about them. We'll also learn about tools such as the OEIS and SageMath.

2. **Binomial coefficients.** In this chapter, we will take a closer and deeper look at binomial coefficients and systematically study their properties as well as some counting problems to which they apply.

3. **The twelvefold way.** This is the problem of counting the ways to distribute (a given number of) balls into (a given number of) boxes. The meaning of this problem depends on

   - whether the balls are labeled or not,
   - whether the boxes are labeled or not,
   - whether we insist that each box gets at least one ball, or at most one ball, or we don't care.

   Thus, this is not 1 problem but 12 different problems. The answers are different, so this will be an opportunity for us to see various objects in combinatorics, such as set partitions or integer partitions.

4. **Permutations.** We will have counted them in Chapter 2 already, but now we will take a closer look at them and study their "inner life".

5. **Lattice paths.** We will count lattice paths satisfying various conditions. This is where will meet the infamous Catalan numbers.

The three main threads of this course are:

- **Counting** – i.e., finding formulas for the sizes of certain finite sets. For example, we will count the permutations of the set $\{1,2,\ldots,n\}$, or the k-element subsets of $\{1,2,\ldots,n\}$ that contain no two consecutive elements. "Count" means finding a formula that expresses the number of such permutations or k-element subsets in terms of n and k.

- **Proving polynomial identities** (such as the binomial formula $(x+y)^n = \sum\limits_{k=0}^{n} \dbinom{n}{k} x^k y^{n-k}$ or various deeper ones).

- **Finding and studying interesting maps between sets.** A basic example of such a map is the "bit-set encoding": the bijective map (= one-to-one correspondence) from

$$\{\text{all subsets of } \{1,2,\ldots,n\}\}$$

to

$$\{\text{all } n\text{-tuples } (i_1,i_2,\ldots,i_n) \in \{0,1\}^n\}$$

(these n-tuples are also known as "length-n bitstrings") that sends each subset S of $\{1,2,\ldots,n\}$ to the bitstring $(i_1,i_2,\ldots,i_n)$, whose entry i_k is 1 if $k \in S$ and 0 otherwise.[1] (Don't worry – we will be more detailed in the actual lectures.)

1. Introduction

Let me begin with some counting questions in no particular order.

1.1. Domino tilings

1.1.1. The problem

Let $n, m \in \mathbb{N}$. Here and in the following, $\mathbb{N}$ means the set $\{0,1,2,\ldots\}$.

[1]Here,

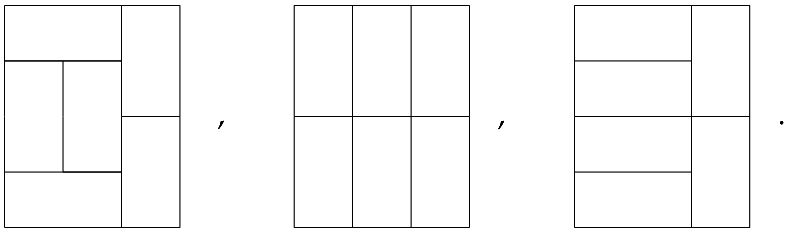
$$\{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n \text{ times}} = \{(i_1,i_2,\ldots,i_n) \mid \text{each } i_k \text{ belongs to } \{0,1\}\}.$$

Let $R_{n,m}$ denote the $n \times m$**-rectangle** – i.e., a rectangle with width n and height m. (We imagine a specific such rectangle drawn somewhere in the plane.)

A **domino** shall mean a 1×2-rectangle or a 2×1-rectangle. More specifically, a **vertical domino** shall mean a 1×2-rectangle, while a **horizontal domino** shall mean a 2×1-rectangle.

A **domino tiling** of $R_{n,m}$ is a way to cover the rectangle $R_{n,m}$ with non-overlapping dominos.

For instance, here are three domino tilings of $R_{3,4}$:

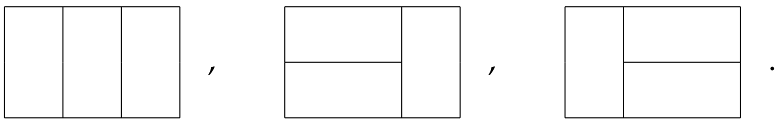


There are several more; can you find a few?

We can now state our first counting problem:

> **Domino tiling counting problem:** How many domino tilings does $R_{n,m}$ have?

For instance, the number of domino tilings of $R_{3,2}$ is 3. Here are all of them:



How can we address larger cases (i.e., larger n or m)?

First, we need to clarify our definition of "domino tiling" by explaining what "a way to cover the rectangle $R_{n,m}$ with non-overlapping dominos" means. What does "cover" mean, and what does "non-overlapping" mean? There are two ways to define these notions:

- **The geometric way:** We really define $R_{n,m}$ as a rectangle of width n and height m in the Euclidean plane[2]. We say that a set of dominos **covers** $R_{n,m}$ if their union (as sets of points) is $R_{n,m}$. It is trickier to define what it means for a set of dominos to be **non-overlapping**. For instance, we can say that it means that their pairwise intersections are 0- or 1-dimensional (or, equivalently, they are just single points or line segments).

  With these definitions, the problem becomes unambiguous and clear. Unfortunately, it is still hard to reason about such tilings, since a lot of visually obvious geometric facts are non-obvious to prove.

[2] For instance, let's say it's the rectangle with corners $(0,0)$, $(n,0)$, $(0,m)$ and (n,m).

- **The combinatorial way:** We redefine $R_{n,m}$ as the set $[n] \times [m]$, where we set
$$[k] := \{1, 2, \ldots, k\} \qquad \text{for each } k \in \mathbb{N}.$$
Its elements are the pairs (i, j) where $i \in [n]$ and $j \in [m]$. We refer to these pairs as **squares**, and we draw them as 1×1-squares in the plane (placing the center of each square (i, j) at the point with Cartesian coordinates (i, j)).

  For instance, here is how $R_{3,4}$ looks like with each square labeled:

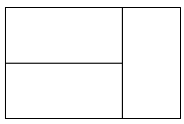
$(1,4)$	$(2,4)$	$(3,4)$
$(1,3)$	$(2,3)$	$(3,3)$
$(1,2)$	$(2,2)$	$(3,2)$
$(1,1)$	$(2,1)$	$(3,1)$

  Thus, $R_{n,m}$ is a **finite** set of size $|R_{n,m}| = nm$. (The **size** of a finite set means the number of its elements, i.e., its cardinality.)

  A **vertical domino** means a set of the form
$$\{(i, j), \ (i, j+1)\} \qquad \text{for some } i, j \in \mathbb{Z}.$$
  A **horizontal domino** means a set of the form
$$\{(i, j), \ (i+1, j)\} \qquad \text{for some } i, j \in \mathbb{Z}.$$
  A **domino** means a vertical domino or a horizontal domino.

  If S is a set of squares (e.g., the rectangle $R_{n,m}$), then a **domino tiling** of S shall mean a set $\{S_1, S_2, \ldots, S_k\}$ of **disjoint** dominos whose union is S (that is, $S_1 \cup S_2 \cup \cdots \cup S_k = S$).

  Of course, we still **draw** our rectangles and dominos as geometric rectangles on a grid, but we are now implementing them as finite sets of pairs of numbers rather than as geometric shapes. For instance, the domino tiling



  is now the set
$$\{\{(1,1), \ (2,1)\}, \quad \{(1,2), \ (2,2)\}, \quad \{(3,1), \ (3,2)\}\}.$$

> This kind of interpretation is known as a **discrete model** – because we're modeling our objects (domino tilings) as finite sets of integers. Note that it is much simpler to reason about than the geometric model above: For example, non-overlappingness of dominos means disjointness of sets in the discrete model, whereas in the geometric model the dominos could meet along an edge.

From now on, we shall always be using the discrete model for our domino tilings – i.e., we define $R_{n,m}$ and domino tilings as in the combinatorial way above.

Now, for any $n, m \in \mathbb{N}$, we define

$$d_{n,m} := (\text{\# of domino tilings of } R_{n,m}) .$$

Here and in the following, the symbol "#" means "number" (or "the number").

Our counting problem thus asks to compute $d_{n,m}$. For instance, we saw that $d_{3,2} = 3$.

In theory, each single $d_{n,m}$ can be computed by brute force: just check all possible sets of dominos for being domino tilings of $R_{n,m}$. This is slow and stupid, but shows that we are dealing with a **finite problem**.

Let us, however, see how to solve this problem more efficiently and (ideally) in greater generality.