Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2022: Homework 5

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due date: **Monday, 2022-12-05** at 11:59 PM on gradescope.
Please solve **3 of the 6 exercises**!

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Prove that

$$p_0(n) + p_1(n) + \cdots + p_k(n) = p_k(n+k).$$

(Recall that $p_i(m)$ denotes the # of partitions of m into i parts.)

1.2 Solution

[...]

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2 Exercise 2

2.1 Problem

Let n and k be two positive integers such that $k > 1$.

Let u be the # of compositions α of n such that each entry of α is congruent[1] to 1 modulo k.

Let w be the # of compositions γ of $n-1$ such that each entry of γ is either 1 or k.

Prove that $u = w$.

2.2 Remark

For instance, if $n = 5$ and $k = 3$, then u counts the compositions

$$(4, 1), \qquad (1, 4), \qquad (1, 1, 1, 1, 1),$$

while w counts the compositions

$$(3, 1), \qquad (1, 3), \qquad (1, 1, 1, 1).$$

If $k = 2$, then both u and w equal the Fibonacci number f_n. (Indeed, $w = f_n$ follows from [Math222, Exercise 2.10.5 **(a)**].)

2.3 Solution

[...]

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Prove that

$$\sum_{\substack{(a_1,a_2,\ldots,a_k) \text{ is a composition} \\ \text{of } n \text{ into } k \text{ parts}}} \frac{1}{a_1 a_2 \cdots a_k} = \frac{k!}{n!} \begin{bmatrix} n \\ k \end{bmatrix}.$$

(Recall that $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the # of permutations $\sigma \in S_n$ with exactly k orbits; it is a Stirling number of the first kind.)

3.2 Hint

Define an *orbit-numbered permutation* to be a permutation $\sigma \in S_n$ whose orbits are numbered by $1, 2, \ldots, j$ in some order (where j is its # of orbits). How many orbit-numbered permutations have k orbits?

(This is not the only approach; there are also non-combinatorial solutions.)

[1] We say that an integer a is *congruent* to an integer b *modulo* an integer k if the difference $a - b$ is a multiple of k. For instance, the even integers are congruent to 0 modulo 2, whereas the odd integers are congruent to 1 modulo 2.

3.3 Solution

[...]

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4 Exercise 4

4.1 Problem

Let X be a finite set. Let A and B be two subsets of X such that $A \cup B = X$ and $A \cap B = \varnothing$. Let σ be a permutation of X. Prove that

$$\left| \sigma\left(A\right) \cap B \right| = \left| \sigma\left(B\right) \cap A \right|.$$

(Here, if C is any subset of X, then $\sigma\left(C\right)$ denotes the set $\left\{\sigma\left(c\right) \mid c \in C\right\}$.)

4.2 Solution

[...]

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Let X be an n-element set, and let i and j be two distinct elements of X. Prove that there are exactly $n!/2$ permutations $\sigma \in S_X$ that satisfy $i \overset{\sigma}{\sim} j$.

(Recall that $i \overset{\sigma}{\sim} j$ means that $i = \sigma^k\left(j\right)$ for some $k \in \mathbb{N}$.)

5.2 Solution

[...]

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. Let X be an n-element set. Let d be the lowest common multiple of the first n positive integers $1, 2, \ldots, n$ (that is, the smallest positive integer that is divisible by each of $1, 2, \ldots, n$).

Let $k \in \mathbb{N}$. Prove that the following two statements are equivalent:

1. We have $\sigma^k = \mathrm{id}$ for each permutation $\sigma \in S_X$.
2. The number k is a multiple of d.

6.2 Remark

For example, if $n = 5$, then $d = \operatorname{lcm}(1, 2, 3, 4, 5) = 60$. Thus, the exercise claims that every permutation σ of a 5-element set satisfies $\sigma^{60} = \mathrm{id}$, and more generally, $\sigma^k = \mathrm{id}$ whenever k is a multiple of 60; and conversely, if k is not a multiple of 60, then some permutation σ does not satisfy $\sigma^k = \mathrm{id}$.

6.3 Solution

[...]

References

[17f-hw7s] Darij Grinberg, *UMN Fall 2017 Math 4990 homework set #7 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw7os.pdf`

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 13 September 2022. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`