Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2022: Homework 4

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due date: **Monday, 2022-11-14** at 11:59 PM on gradescope.
Please solve **3 of the 6 exercises**!

1 Exercise 1

1.1 Problem

Let A, B and C be three finite sets such that $C \subseteq B$. Let $a = |A|$ and $b = |B|$ and $c = |C|$.
Prove that the # of maps $f : A \to B$ satisfying[1] $C \subseteq f(A)$ is

$$\sum_{k=0}^{c} (-1)^k \binom{c}{k} (b-k)^a .$$

1.2 Remark

Applying this to $C = B$, we recover the explicit formula for $\operatorname{sur}(a, b)$ ([Math222, Theorem 2.4.17]), because a map $f : A \to B$ satisfying $B \subseteq f(A)$ is the same as a surjection from A to B.

1.3 Solution

[...]

[1] We let $f(A)$ denote the range of f, that is, the set $\{f(x) \mid x \in A\}$.

2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} (-2)^{n-k} \binom{n}{k} \binom{2k}{k} = \binom{n}{n/2}.$$

[Note that the right hand side is 0 when n is odd.]

2.2 Hint

Consider $2 \times n$-matrices $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$ with the following properties:

1. All entries of A belong to the set $\{0, 1\}$.
2. We have $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$ (that is, the entries in each row of A add up to the same number).
3. We have $a_i + b_i = 1$ for all $i \in [n]$ (that is, each column of A contains a 0 and a 1).

Count these matrices in two different ways: once directly, and once using the Principle of Inclusion and Exclusion (letting U be the set of matrices satisfying properties 1 and 2).

2.3 Solution

[...]

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. Let U be a finite set. Let $A_1, A_2, \ldots, A_n$ be n subsets of U. Let $k \in \mathbb{N}$. Let G_k be the set of all elements of U that belong to at least k of the n subsets $A_1, A_2, \ldots, A_n$. In other words, let

$$G_k = \{s \in U \mid \text{the number of } i \in [n] \text{ satisfying } s \in A_i \text{ is } \geq k\}.$$

Prove that

$$|G_k| = \sum_{I \subseteq [n]} (-1)^{|I|-k} \binom{|I|-1}{|I|-k} \left| \bigcap_{i \in I} A_i \right|.$$

Here, the intersection $\bigcap_{i \in \varnothing} A_i$ is understood to mean the whole set U.

3.2 Remark

Note that $G_0 = U$ and $G_1 = A_1 \cup A_2 \cup \cdots \cup A_n$. Thus, by setting $k = 1$ in this exercise, you recover the Principle of Inclusion and Exclusion (in the union form).

3.3 Hint

This is an analogue of [17f-hw7s, Exercise 1], which you can use without proof. You can also use [Math222, Exercise 2.1.1].

3.4 Solution

[...]

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4 Exercise 4

4.1 Problem

Let U, V and W be three sets, with U and W being finite. Prove that

$$\sum_{I \subseteq U} (-1)^{|(I \setminus V) \cup W|} = \begin{cases} 2^{|U|} (-1)^{|W|}, & \text{if } U \subseteq V \cup W; \\ 0, & \text{otherwise.} \end{cases}$$

4.2 Hint

This equality is in the vein of [Math222, Section 2.9.7].

4.3 Solution

[...]

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5 Exercise 5

5.1 Problem

Let n and k be two positive integers with $k \geq 2$.

A composition $(a_1, a_2, \ldots, a_k)$ of n into k parts will be called *lopsided* if one of its entries a_i is larger than $n/2$. (For instance, the composition $(1, 5, 3)$ is lopsided, but the composition $(1, 4, 3)$ is not.)

Prove that the # of lopsided compositions of n into k parts is $k \dbinom{\lceil n/2 \rceil - 1}{k-1}$.

5.2 Remark

This can be restated as follows: If a stick of length n is broken into k pieces at random (by choosing a random $(k-1)$-element subset of $[n-1]$ and breaking the stick at the positions in this subset), then the probability that these k pieces are the sidelengths of a (possibly degenerate) k-gon is $1 - \dfrac{k\dbinom{\lceil n/2 \rceil - 1}{k-1}}{\dbinom{n-1}{k-1}}$. (Indeed, k positive real numbers $a_1, a_2, \ldots, a_k$ are the sidelengths of a (possibly degenerate) k-gon if and only if they satisfy the "polygon inequalities", which say that no a_i is larger than $\dfrac{a_1 + a_2 + \cdots + a_k}{2}$.)

5.3 Solution

[...]

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. How many (ordered) pairs (A, B) of two nonempty subsets of $[n]$ have the property that $\max A > |B|$ and $\max B > |A|$?

[**Example:** The pair $(\{1,4\}, \{1,2,3\})$ (for $n = 4$) qualifies, since $\max\{1,4\} = 4 > 3 = |\{1,2,3\}|$ and $\max\{1,2,3\} = 3 > 2 = |\{1,4\}|$. However, the pair $(\{1,3\}, \{2,3,4\})$ does not qualify, since $\max\{1,3\} \leq |\{2,3,4\}|$.]

6.2 Solution

[...]

References

[17f-hw7s] Darij Grinberg, *UMN Fall 2017 Math 4990 homework set #7 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw7os.pdf`

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 13 September 2022. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`