Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2022: Homework 3

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due date: **Friday, 2022-10-21** at 11:59 PM on gradescope.
Please solve **3 of the 6 exercises**!

1 Exercise 1

1.1 Problem

Recall the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ (with $f_0 = 0$ and $f_1 = 1$). Prove that

$$f_{n+1} = 1 + \sum_{k=2}^{n} (-1)^k \binom{n}{k} f_{k-1} \qquad \text{for each } n \in \mathbb{N}.$$

1.2 Solution

[...]

2 Exercise 2

2.1 Problem

Let[1] $n \in \mathbb{N}$ and $m \in \mathbb{R}$. Let $p_1, p_2, \ldots, p_n \in \mathbb{R}$ be fixed. Prove that

$$\sum_{\substack{(a_1,a_2,\ldots,a_n)\in\mathbb{N}^n;\\ a_1+a_2+\cdots+a_n=m}} \binom{p_1}{a_1}\binom{p_2}{a_2}\cdots\binom{p_n}{a_n} = \binom{p_1+p_2+\cdots+p_n}{m}.$$

2.2 Hint

You are allowed to use the Chu–Vandermonde identity [Math222, Theorem 2.6.1].

2.3 Solution

[...]

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. Let A denote the $n \times n$-matrix

$$\left(\binom{i}{j}\right)_{0\le i,j\le n-1} = \begin{pmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{n-1} \\ \binom{1}{0} & \binom{1}{1} & \cdots & \binom{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{0} & \binom{n-1}{1} & \cdots & \binom{n-1}{n-1} \end{pmatrix} \in \mathbb{Q}^{n\times n}.$$

Note that the rows and the columns of this matrix are indexed by the numbers $0, 1, \ldots, n-1$ rather than by the usual numbers $1, 2, \ldots, n$. (For example, if $n = 4$, then $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}$. So the matrix A is a piece of Pascal's triangle, including some of the zeroes to its right.)

Find and prove a formula for each entry of the inverse matrix A^{-1}.

3.2 Hint

Recall that the inverse matrix A^{-1} is the $n \times n$-matrix B satisfying $AB = BA = I_n$, where I_n denotes the $n \times n$-identity matrix. By a known result in linear algebra, it suffices to prove one of the two equalities $AB = I_n$ and $BA = I_n$; the other then follows automatically.

[1] Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

In SageMath, matrices can be inputted in many ways, e.g., using lists of lists: The matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in \mathbb{Q}^{2\times 2}$ is written `Matrix(QQ, [[1, 2], [3, 4]])` . The easiest way to get the inverse of a matrix A is to type `~A` (don't copy-paste; use the tilde key on your keyboard).

3.3 Solution

[...]

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Consider $\dbinom{x}{n} = \dfrac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$ as a polynomial in x (with rational coefficients). Prove that the derivative of this polynomial is

$$\frac{d}{dx}\binom{x}{n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{x}{n-k}.$$

4.2 Remark

For example, for $n = 3$, this is saying that

$$\frac{d}{dx}\binom{x}{3} = \frac{1}{1}\binom{x}{2} - \frac{1}{2}\binom{x}{1} + \frac{1}{3}\binom{x}{0},$$

which can be easily checked.

You are free to use the fact that a polynomial with rational coefficients is uniquely determined by its values at all real numbers (or even at infinitely many given real numbers; see [Math222, Corollary 2.6.9] for details).

4.3 Solution

[...]

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$ be such that the numbers $a, a+1, \ldots, a+n$ are nonzero. Prove that

$$\sum_{k=0}^{n} \frac{(-1)^k}{a+k}\binom{n}{k} = \frac{1}{a\dbinom{n+a}{n}}.$$

5.2 Solution

[...]

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. Let A be the same $n \times n$-matrix as in Exercise 2. Let B be the matrix AA^T, where A^T denotes the transpose of A. Find and prove an explicit formula for each entry of B.

6.2 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 13 September 2022. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`