Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2022: Homework 2

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due date: **Wednesday, 2022-10-14** at 11:59 PM on gradescope.
Please solve **3 of the 7 exercises**!

1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\prod_{k=0}^{n} \binom{n}{k} = \prod_{k=1}^{n-1} \binom{n}{k} = \frac{n!^{n+1}}{\left(\prod\limits_{k=0}^{n} k!\right)^2} = \prod_{j=1}^{n} j^{2j-n-1} = \frac{\prod\limits_{j=1}^{n} j^{2j}}{(n!)^{n+1}}.$$

1.2 Solution

[...]

2 Exercise 2

2.1 Problem

Let $a, b, n, m \in \mathbb{R}$. Prove that

$$\binom{n}{m-a}\binom{n+a}{m-b}\binom{n+b}{m} = \binom{n}{m-b}\binom{n+b}{m-a}\binom{n+a}{m}.$$

2.2 Hint

Be aware of the assumptions of the results you are using! Not every formula can be used in every case. Some case distinction is probably necessary.

2.3 Solution

[...]

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3 Exercise 3

3.1 Problem

Let S be an n-element set. Show that $\sum\limits_{A\subseteq S} \sum\limits_{B\subseteq S} |A \cap B| = n \cdot 4^{n-1}$.

3.2 Solution

[...]

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4 Exercise 4

4.1 Problem

For any $n \in \mathbb{N}$, let a_n be the # of subsets S of $[n]$ satisfying $3 \mid \sum\limits_{s\in S} s$.

(a) Prove that $a_n = 2a_{n-3} + 2^{n-2}$ for any $n \geq 3$.

(b) Prove that $a_n = 2a_{n-1} - [3 \mid n-1] \cdot 2^{(n-1)/3}$ for any $n \geq 1$.

4.2 Hint

Something similar appeared in [18f-hw2s, Exercise 3 **(a)**] (but here, it is the sum of the elements of S, rather than their number, that has to be a multiple of 3).

4.3 Solution

[...]

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Two elements x and y of $[2n]$ are said to be *antipodes* if $x + y = 2n + 1$.

How many subsets S of $[2n]$ contain no two antipodes?

5.2 Solution

[...]

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. Give a closed-form expression (no summation signs) for $\sum_{k=0}^{n} k \binom{n}{2k}$.

6.2 Solution

[...]

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7 Exercise 7

7.1 Problem

Consider the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$.

Let $n \geq 2$ be an integer. A subset S of $[n]$ is said to be *cyc-lacunar* if it is lacunar and does not satisfy $\{1, n\} \subseteq S$. (Visually speaking, this condition says that if the numbers $1, 2, \ldots, n$ are drawn on a circle in this order, then S contains no two adjacent numbers.)

(a) Prove that the # of cyc-lacunar subsets of $[n]$ is $f_{n+2} - f_{n-2}$.

(b) Prove that the # of cyc-lacunar subsets of $[n]$ is $f_{n+1} + f_{n-1}$.

7.2 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 13 September 2022. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`

[18f-hw2s] Darij Grinberg, *UMN Fall 2018 Math 5705 homework set #2 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/18f/hw2s.pdf`