Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2022: Homework 1

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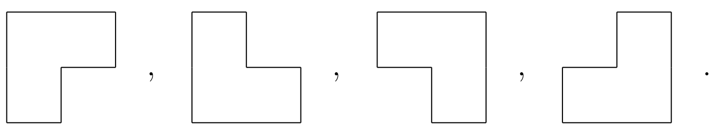
due date: **Monday, 2022-10-03** at 11:59 PM on gradescope.
Please solve **3 of the 6 exercises**!

1 Exercise 1

1.1 Problem

Let us define a slight variation on domino tilings. We shall use the notations of [Math222, §1.1].

An *L-tromino* will mean a set of squares that has one of the four forms



(Formally speaking, it is a set of the form $\{(i,j), (i',j), (i,j')\}$, where $i, j \in \mathbb{Z}$ and $i' \in \{i-1, i+1\}$ and $j' \in \{j-1, j+1\}$.)

If S is a set of squares, then an *L-tromino tiling* of S will mean a set of disjoint L-trominos whose union is S.

For any $n \in \mathbb{N}$, we let M_n denote the number of L-tromino tilings of the rectangle $R_{n,3}$.

We shall use the Iverson bracket notation[1].

Prove that

$$M_n = [2 \mid n] \cdot 2^{n/2} \qquad \text{for each } n \in \mathbb{N}.$$

[1] This means the following:

If $\mathcal{A}$ is any statement (such as "$1+1=2$" or "$1+1=1$" or "there exist infinitely many primes"), then

1.2 Hint

Feel free to take inspiration from the solution to [19fco-hw1s, Exercise 2], but make sure to explain why every L-tromino tiling of $R_{n,3}$ fits within your mold.

1.3 Solution

[...]

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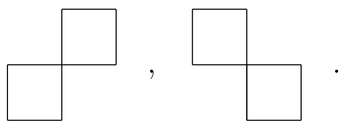
2 Exercise 2

2.1 Problem

Again, we shall use the notations of [Math222, §1.1].

A *diagomino* will mean a set of squares that has one of the two forms



(Formally speaking, it is a set of the form $\{(i,j),(i+1,j')\}$, where $i,j \in \mathbb{Z}$ and $j' \in \{j-1,j+1\}$. Nevermind that such a tile wouldn't hold together in real life...)

If S is a set of squares, then a *diagomino tiling* of S will mean a set of disjoint diagominos whose union is S.

For any $n \in \mathbb{N}$, find

(a) the number of diagomino tilings of the rectangle $R_{n,2}$.

(b) the number of diagomino tilings of the rectangle $R_{n,3}$.

2.2 Solution

[...]

$[\mathcal{A}]$ stands for the number

$$\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$$

This number belongs to $\{0,1\}$, and is called the *truth value* of $\mathcal{A}$. For example,

$$[1+1=2]=1, \qquad [1+1=1]=0, \qquad [\text{there exist infinitely many primes}]=1.$$

3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. How many 6-tuples (A, B, C, D, E, F) of subsets of $[n]$ satisfy $A \cup B \cup C = D \cup E \cup F$?

3.2 Hint

Similar problems appear (with solutions) in [19fco-hw2s, Exercise 2], [17f-hw3s, Exercise 1] and [Math235, Exercise 7.3.1].

3.3 Solution

[...]

4 Exercise 4

4.1 Problem

[This exercise was removed.]

4.2 Solution

[...]

5 Exercise 5

5.1 Problem

A finite set S of integers is said to be *Schreier* if it is nonempty and satisfies $\min S \geq |S|$.

Let $n \in \mathbb{N}$. Count (i.e., find the # of all) the Schreier subsets of $[n]$.

[**Example:** The Schreier subsets of $[4]$ are

$$\{1\}, \ \{2\}, \ \{3\}, \ \{4\}, \ \{2,3\}, \ \{2,4\}, \ \{3,4\}.$$

So there are 7 of them.]

5.2 Solution

[...]

6 Exercise 6

6.1 Problem

Consider the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$. Prove that any $n \geq 2$ satisfies

$$f_n^2 = f_{n+1} f_{n-1} - (-1)^n \qquad \text{and} \qquad f_n^4 = f_{n+2} f_{n+1} f_{n-1} f_{n-2} + 1.$$

6.2 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 13 September 2022. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`

[Math235] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 22 March 2021. `http://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`

[17f-hw3s] Darij Grinberg, *UMN Fall 2017 Math 4990 homework set #3 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw3os.pdf`

[19fco-hw0s] Darij Grinberg, *Drexel Fall 2019 Math 222 homework set #0 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/hw0s.pdf`

[19fco-hw1s] Darij Grinberg, *Drexel Fall 2019 Math 222 homework set #1 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/hw1s.pdf`

[19fco-hw2s] Darij Grinberg, *Drexel Fall 2019 Math 222 homework set #2 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/hw2s.pdf`