Drexel University, Department of Mathematics

Math 533: Abstract Algebra I, Winter 2021: Homework 0

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January 6, 2021

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1 Exercise 1

1.1 Problem

How many semesters (or quarters) of abstract algebra have you taken (include Galois theory, representation theory, group theory)?

1.2 Solution

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2 Exercise 2

2.1 Problem

How familiar are you with the notions of

1. ring;
2. normal subgroup of a group;
3. Jordan canonical form (aka Jordan normal form) of a matrix;

4. quotient vector space V/W;
5. exact sequence;
6. determinant;
7. Cayley–Hamilton theorem;
8. greatest common divisor of two (univariate) polynomials;
9. complex number;
10. Gaussian integer;
11. primitive n-th root of unity;
12. discrete Fourier transform?

(Write in a number between 0 (for "never seen it") and 5 (for "remember the important properties and could recall their proofs in 15 minutes without looking up") for each one.)

2.2 Solution

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3 Exercise 3

3.1 Problem

(a) Factor the polynomial $a^3 + b^3 + c^3 - 3abc$.

(b) Factor the polynomial $bc\,(b-c) + ca\,(c-a) + ab\,(a-b)$.

(c) How general have your methods been? Did you use tricks specific to the given polynomials, or do you have an algorithm for factoring any polynomial (say, with integer coefficients)?

3.2 Solution

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4 Exercise 4

4.1 Problem

Simplify $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Let $a_1, a_2, \ldots, a_n$ be n integers, and let $b_1, b_2, \ldots, b_n$ be n further integers. The Gaussian elimination tells you how to solve the system

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0;$$
$$b_1 x_1 + b_2 x_2 + \cdots + b_n x_n = 0$$

for n unknowns $x_1, x_2, \ldots, x_n \in \mathbb{Q}$. The answer, in general, will have the form "all $\mathbb{Q}$-linear combinations (i.e., linear combinations with rational coefficients) of a certain bunch of vectors". (More precisely, "a certain bunch of vectors" are $n-2$ or $n-1$ or n vectors with rational coordinates, depending on the rank of the $2 \times n$-matrix $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$.)

Now, how can you solve the above system for n unknowns $x_1, x_2, \ldots, x_n \in \mathbb{Z}$? Will the answer still be "all $\mathbb{Z}$-linear combinations (i.e., linear combinations with integer coefficients) of a certain bunch of vectors"?

What about more general systems of linear equations to be solved for integer unknowns?

5.2 Solution

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6 Exercise 6

6.1 Problem

You are given a 4×4-grid of lamps, each of which is either on or off. For example, writing 1 for "on" and 0 for "off", it may look as follows:

1	0	0	1
1	1	0	0
1	0	0	1
0	1	1	1

In a single move, you can toggle any lamp (i.e., turn it on if it was off, or turn it off if it was on); however, this will also toggle every lamp adjacent to it. ("Adjacent to it" means "having a grid edge in common with it"; thus, a lamp will have 2 or 3 or 4 adjacent lamps.)

For example, if we toggle the second lamp (from the left) in the topmost row in the above example grid, then we obtain

0	**1**	**1**	1
1	**0**	0	0
1	0	0	1
0	1	1	1

(where the boldfaced numbers correspond to the lamps that have been affected by the move).

Assume that all lamps are initially off. Can you (by a strategically chosen sequence of moves) achieve a state in which all lamps are on?

[*Remark:* You can play this game (albeit with a 5×5-grid) on `https://codepen.io/wintlu/pen/ZJJLGz` .]

6.2 Solution

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7 Exercise 7

7.1 Problem

(a) How many of the numbers $0, 1, \ldots, 6$ appear as remainders of a perfect square divided by 7 ?

(b) How many of the numbers $0, 1, \ldots, 13$ appear as remainders of a perfect square divided by 14 ?

What about replacing 7 or 14 by n? Can you do better than just squaring them all?

[For example, 3 of the numbers $0, 1, \ldots, 4$ appear as remainders of a perfect square divided by 5 – namely, the three numbers $0, 1, 4$.]

7.2 Solution

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8 Exercise 8

8.1 Problem

Solve the following system of equations:

$$a^2 + b + c = 1;$$
$$b^2 + c + a = 1;$$
$$c^2 + a + b = 1$$

for three complex numbers a, b, c.

8.2 Solution

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9 Exercise 9

9.1 Problem

The following triangular table shows the binomial coefficients $\binom{n}{m}$ for $n \in \{0, 1, \ldots, 7\}$ and $m \in \{0, 1, \ldots, n\}$:

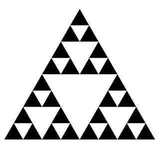
									$k=0$ $\swarrow$							
$n=0 \rightarrow$								1		$k=1$ $\swarrow$						
$n=1 \rightarrow$							1		1		$k=2$ $\swarrow$					
$n=2 \rightarrow$						1		2		1		$k=3$ $\swarrow$				
$n=3 \rightarrow$					1		3		3		1		$k=4$ $\swarrow$			
$n=4 \rightarrow$				1		4		6		4		1		$k=5$ $\swarrow$		
$n=5 \rightarrow$			1		5		10		10		5		1		$k=6$ $\swarrow$	
$n=6 \rightarrow$		1		6		15		20		15		6		1		$k=7$ $\swarrow$
$n=7 \rightarrow$	1		7		21		35		35		21		7		1	

(This is part of what is known as *Pascal's triangle.*)

Now, in this table, let us replace each even number by a 0 and each odd number by a 1. We obtain

									$k=0$ $\swarrow$							
$n=0 \rightarrow$								1		$k=1$ $\swarrow$						
$n=1 \rightarrow$							1		1		$k=2$ $\swarrow$					
$n=2 \rightarrow$						1		0		1		$k=3$ $\swarrow$				
$n=3 \rightarrow$					1		1		1		1		$k=4$ $\swarrow$			
$n=4 \rightarrow$				1		0		0		0		1		$k=5$ $\swarrow$		
$n=5 \rightarrow$			1		1		0		0		1		1		$k=6$ $\swarrow$	
$n=6 \rightarrow$		1		0		1		0		1		0		1		$k=7$ $\swarrow$
$n=7 \rightarrow$	1		1		1		1		1		1		1		1	

This looks rather similar to the third evolutionary stage of Sierpinski's triangle:



(Each 0 in the above table corresponds to a white △ triangle, and each 1 corresponds to a black ▲ triangle.)

Where does this similarity come from?

9.2 Solution

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10 Exercise 10

10.1 Problem

A *conic* means a curve of the form

$$\left\{(x,y) \in \mathbb{R}^2 \mid ax^2 + bxy + cy^2 + dx + ey + f = 0\right\},$$

where a, b, c, d, e, f are six real numbers such that $(a, b, c, d, e, f) \neq (0, 0, 0, 0, 0, 0)$. Examples of conics are

- any circle, e.g., the unit circle $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$;
- more generally, any ellipse;
- any parabola, e.g., $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y = 0\}$;
- any hyperbola, e.g., $\{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$ or $\{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$;
- the union of any two lines, e.g., $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$.

A conic is said to be *nondegenerate* if it is not the union of two lines.

(a) What is the maximum number of points in which a nondegenerate conic can intersect a line?

(b) What is the maximum number of points in which two nondegenerate conics can intersect each other?

10.2 Solution

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References