Math 220 Fall 2021, Lecture 24: Relations and functions

1. Relations and functions

1.1. Defining functions (cont'd)

Recall our provisional definition of a function:

> **Definition 1.1.1** (provisional definition of a function)**.** Let X and Y be two sets. A **function** from X to Y is a rule that transforms each element of X into some element of Y. If we call this function f, then the result of applying it to an $x \in X$ will be called $f(x)$ (or sometimes fx).

And then our rigorous definition of a function:

> **Definition 1.1.2** (rigorous definition of a function)**.** Let X and Y be two sets. A **function** from X to Y means a relation R between X and Y that has the following property: For each $x \in X$, there exists **exactly one** $y \in Y$ such that $x \; R \; y$.

In what sense do these two definitions define the same thing? What do the functions in the sense of the provisional definition have to do with the functions in the sense of the rigorous definition?

To answer this, let me refer to the functions in the sense of the provisional definition as **provisional functions**, and to the functions in the sense of the rigorous definition as **rigorous functions**. So we have to explain:

1. In what sense is a provisional function a rigorous function?

2. In what sense is a rigorous function a provisional function?

The key is that a rigorous function is a relation that relates each $x \in X$ to exactly one $y \in Y$, so we can call the latter y the **value** of the function on x, and then treat our function as the rule that transforms each x into the corresponding value y. In more detail:

Answer to question 2: If we are given a rigorous function R, then we can consider the rule that transforms each $x \in X$ into the unique $y \in Y$ such that $x \; R \; y$; this latter rule is a provisional function from X to Y.

Answer to question 1: If f is a provisional function from X to Y, then we can consider the relation R between X and Y that is given by

$$(x \; R \; y) \Longleftrightarrow (y = f(x)).$$

This relation is a rigorous function from X to Y.

So we can translate rigorous functions and provisional functions into one another. Thus, we can think of the two concepts as being the same thing. In particular, all the notations we introduced for provisional functions will be applied to rigorous functions as well.

1.2. Examples of functions

Example 1.2.1. Consider the function $f_1 : \{1,2,3,4\} \to \{1,2,3,4\}$ that sends $1,2,3,4$ to $3,2,3,3$, respectively. As a rigorous function, it is the relation R that satisfies

$$1 \, R \, 3, \qquad 2 \, R \, 2, \qquad 3 \, R \, 3, \qquad 4 \, R \, 3$$

and nothing else. In other words, it is the relation

$$\{(1,3), (2,2), (3,3), (4,3)\}.$$

Example 1.2.2. (a) Consider the function

$$f_2 : \{1,2,3,\ldots\} \to \{1,2,3,\ldots\}, \qquad n \mapsto (\text{the number of positive divisors of } n).$$

As a relation, it is

$$\{(1,1), (2,2), (3,2), (4,3), (5,2), (6,4), (7,2), (8,4), (9,3), (10,4), \ldots\}$$

(we cannot list all the pairs, since there are infinitely many).

(b) What about the function

$$\widetilde{f_2} : \mathbb{Z} \to \{1,2,3,\ldots\}, \qquad n \mapsto (\text{the number of positive divisors of } n) \ ?$$

There is no such function. Indeed, $\widetilde{f_2}(0)$ would have to be ∞, since 0 has infinitely many positive divisors; but $\infty \notin \{1,2,3,\ldots\}$.

(c) What about the function

$$f_3 : \{1,2,3,\ldots\} \to \{1,2,3,\ldots\}, \qquad n \mapsto (\text{the smallest prime divisor of } n) \ ?$$

Again, there is no such function, because $f_3(1)$ makes no sense (the number 1 has no prime divisors and thus no smallest prime divisor).

(d) What about the function

$$f_4 : \mathbb{Q} \to \mathbb{Z}, \qquad \frac{a}{b} \mapsto a \ ?$$

This, too, is not an actual function. Indeed, a rational number can be written as $\frac{a}{b}$ for **several different** pairs of integers (a,b); these different pairs would lead to different values of a. So if this function f_4 would exist, then we would have $f_4\left(\frac{2}{3}\right) = 2$ and $f_4\left(\frac{4}{6}\right) = 4$, which would lead to $2 = 4$ (because $\frac{2}{3} = \frac{4}{6}$).

The issues we have just discussed (functions being misdefined either because their values make no sense, or because their values don't lie in Y, or because their values are ambiguous) are known as **well-definedness** issues; one often says that "a function is well-defined" to say that its definition has no such issues (i.e., its definition really defines a function).

For example, the function

$$\begin{aligned} f_4 : \mathbb{Q} &\to \mathbb{Z}, \\ \frac{a}{b} &\mapsto a \end{aligned}$$

is not well-defined (by which we mean that there is no such a function), but the function

$$\begin{aligned} f_5 : \mathbb{Q} &\to \mathbb{Q}, \\ \frac{a}{b} &\mapsto \frac{a^2}{b^2} \end{aligned}$$

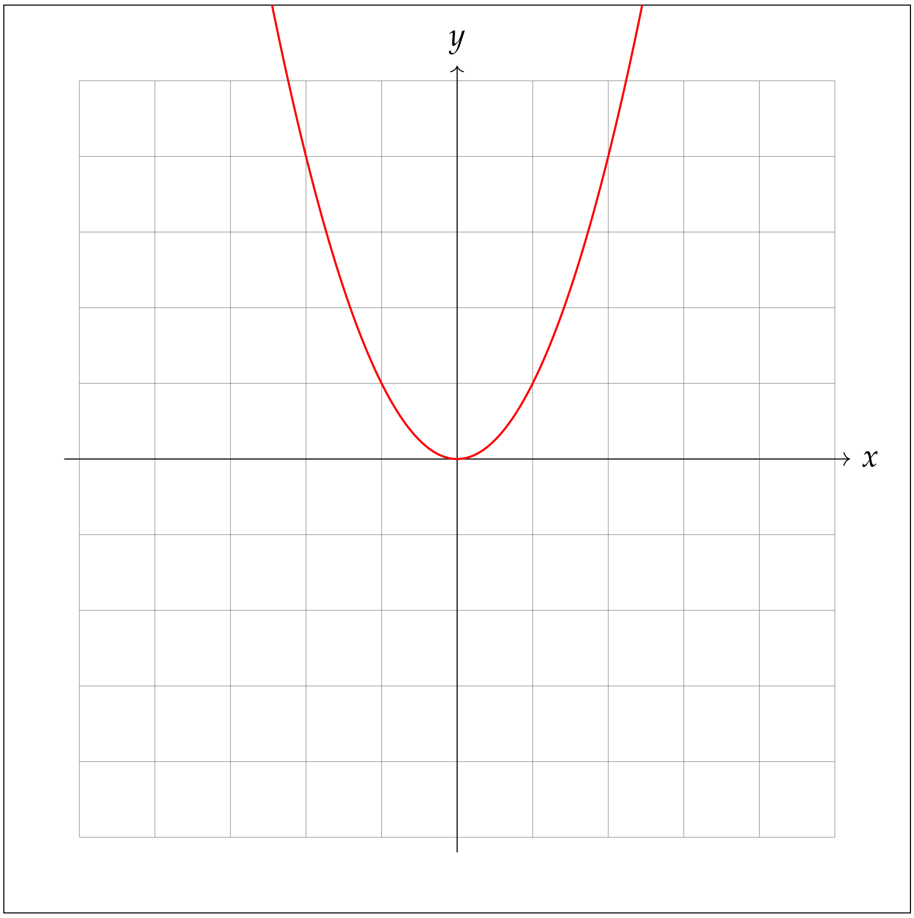
is well-defined, because if you write a given rational number as $\frac{a}{b}$ for different pairs (a, b), then the resulting fractions $\frac{a^2}{b^2}$ will all be equal.

Example 1.2.3. For any set X, there is an **identity function** $\mathrm{id}_X : X \to X$. This is the function that sends each element $x \in X$ to x itself.

A function $f : X \to Y$ can be described either by a rule or by a list of values (if X is finite) or as a relation. It can also be visually represented by a graph (= plot) if $X = Y = \mathbb{R}$. For example, the "take the square" function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto x^2 \end{aligned}$$

has the plot



(This curve is known as a parabola.)

1.3. Composition of functions

There are some ways to transform functions into other functions. The most important such way is **composition**:

Definition 1.3.1. Let X, Y and Z be three sets, and let $f : Y \to Z$ and $g : X \to Y$ be two functions. Then, $f \circ g$ denotes the function

$$X \to Z,$$
$$x \mapsto f(g(x)).$$

In other words, $f \circ g$ is the function that first applies g and then applies f. This function $f \circ g$ is called the **composition** of f with g.

In terms of relations, if we view f and g as relations F and G, then $f \circ g$ is the relation

$$\{(x,z) \mid \text{there exists } y \in Y \text{ such that } x \, G \, y \text{ and } y \, F \, z\}.$$

Example 1.3.2. Let

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto \sin x - \cos x. \end{aligned}$$

Let

$$\begin{aligned} g : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto x^2 + x^3. \end{aligned}$$

Then, for all $x \in \mathbb{R}$, we have

$$(f \circ g)(x) = f(g(x)) = f\left(x^2 + x^3\right) = \sin\left(x^2 + x^3\right) - \cos\left(x^2 + x^3\right)$$

and

$$(g \circ f)(x) = g(f(x)) = g(\sin x - \cos x) = (\sin x - \cos x)^2 + (\sin x - \cos x)^3.$$

On examples like this, you can easily see that $f \circ g$ and $g \circ f$ are usually different functions (if they are both defined to begin with). So composition of functions is not commutative. However, it does have other properties:

Theorem 1.3.3 (associativity of composition)**.** Let $f : Z \to W$, $g : Y \to Z$ and $h : X \to Y$ be three functions. Then,

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Proof. Both functions $f \circ (g \circ h)$ and $(f \circ g) \circ h$ operate on a given element $x \in X$ in the same way: First apply h, then apply g, then apply f. Formally speaking: For each $x \in X$, we have

$$\begin{aligned} &(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x))) \qquad \text{and} \\ &((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))), \end{aligned}$$

so that

$$(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x).$$

But two functions that operate on each $x \in X$ in the same way are equal. Thus, we conclude that

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

□

Theorem 1.3.4. Let $f : X \to Y$ be a function. Then,

$$f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f.$$

Proof. Straightforward. □