Math 220 Fall 2021, Lecture 23: Relations and functions

1. Relations and functions

1.1. Introduction to functions

One of the main notions in mathematics is that of a **function**, aka **map**, aka **mapping**, aka **transformation**.

You surely have seen some functions in high school. E.g., you may remember that there is a function "$f(t) = t^2$", or to be precise: There is a function f that sends every real number t to its square (that is, t^2).

So what is a function?

Based on such examples, it should be some kind of rule that transforms numbers into other numbers. But we don't need to restrict ourselves to numbers; we can just as well have it transform any sort of objects into any further sort of objects.

So here is a first provisional definition of a function:

Definition 1.1.1 (provisional definition of a function)**.** Let X and Y be two sets. A **function** from X to Y is a rule that transforms each element of X into some element of Y. If we call this function f, then the result of applying it to an $x \in X$ will be called $f(x)$ (or sometimes fx).

Some comments on this:

- The function has to "work" for each element of X. It cannot decline to operate on some elements! Thus, "take the reciprocal" is not a well-defined function from $\mathbb{R}$ to $\mathbb{R}$, since it does not operate on 0 (there is no reciprocal of 0).

- The function cannot be ambiguous. So "take some power of the number" is not a function from $\mathbb{R}$ to $\mathbb{R}$, because different powers give different results. There **is** a "multi-valued" variant of functions around, but those aren't called functions; they are a kind of relations, which we will also cover very soon.

- We write "$f : X \to Y$" for "f is a function from X to Y".

- Instead of "f transforms x into y", we often say "f sends x to y" or "f takes the value y at x" or "y is the value of f at x" or "y is the image of x under f" or "applying f to x yields y" or "$f : x \mapsto y$".

  For instance, if f is the "take the square" function from $\mathbb{R}$ to $\mathbb{R}$, then $f(2) = 4$, so that f sends 2 to 4, or takes the value 4 at 2, etc., or $f : 2 \mapsto 4$.

- The notation

$$\begin{aligned} X &\to Y, \\ x &\mapsto (\text{some expression involving } x) \end{aligned}$$

(for example, the expression can be something like x^5 or $\frac{1}{x+4}$ or $\frac{x}{x+1}$) is shorthand for "the function from X to Y that sends each element x of X to the expression given on the right hand side".

For example,

$$\mathbb{R} \to \mathbb{R},$$
$$x \mapsto x^2$$

is the "take the square" function. For another example,

$$\mathbb{R} \to \mathbb{R},$$
$$x \mapsto \frac{x}{\sin x + 15}$$

is the function that takes the sine, then adds 15, then divides the input value by the result. (Note that the denominator $\sin x + 15$ is never 0, because $-1 \leq \sin x$ for all $x \in \mathbb{R}$. This ensures that the expression $\frac{x}{\sin x + 15}$ is always meaningful, and thus we really have a function in front of us.) For yet another example,

$$\mathbb{R} \to \mathbb{R},$$
$$x \mapsto 2$$

is the function that sends every number to 2; it is an example of a constant function. (Yes, "expression involving x" does not require x to actually appear in the expression.) For yet another example,

$$\mathbb{Z} \to \mathbb{Q},$$
$$x \mapsto 2^x$$

is another function, and here are some of its values:

x	-2	-1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

- The notation

$$f : X \to Y,$$
$$x \mapsto (\text{some expression involving } x)$$

means that we take the function from X to Y that sends each $x \in X$ to the expression on the right hand side, and we call this function f (or, if f is already defined, we claim that this is the function f).

- If the set X is finite, then a function $f : X \to Y$ can be specified by simply listing all its values. For example, I can define a function $h : \{0, 2, 4\} \to \mathbb{N}$ as follows:

x	0	2	4
$h(x)$	92	20	92

  The values here have been chosen randomly, for no particular reason. A function does not have to be "natural" or "meaningful" in any common-sense interpretation of these words; all it has to do is transform each element of X into some element of Y.

- If f is a function from X to Y, then the sets X and Y are part of the function. Thus,

$$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Q}, \\ x &\mapsto 2^x \end{aligned}$$

  and

$$\begin{aligned} g : \mathbb{N} &\to \mathbb{Q}, \\ x &\mapsto 2^x \end{aligned}$$

  and

$$\begin{aligned} h : \mathbb{N} &\to \mathbb{N}, \\ x &\mapsto 2^x \end{aligned}$$

  are three distinct functions! It is worth distinguishing between them, even though they are given by the same expression, because e.g., the functions g and h only take integer values, whereas the function f takes some non-integer values as well. Even g and h should not be regarded as the same.

  So when are two functions equal? Two functions $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are **equal** if and only if

$$X_1 = X_2 \qquad \text{and} \qquad Y_1 = Y_2 \qquad \text{and} \qquad f_1(x) = f_2(x) \text{ for all } x \in X_1.$$

  An example of two equal functions is

$$\begin{aligned} f_1 : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto (\sin x)^2 \end{aligned}$$

  and

$$\begin{aligned} f_2 : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto 1 - (\cos x)^2, \end{aligned}$$

  since each $x \in \mathbb{R}$ satisfies $(\sin x)^2 = 1 - (\cos x)^2$.

So we have a good idea of what a function is, but we don't have a formal definition at this point, which makes it hard to argue about functions $f : X \to Y$ that can neither be given by an explicit formula (such as "take the square") nor be given by a complete list of values (e.g., because X is infinite). So we are looking for some formal definition. The provisional definition above does not qualify, since it reduces the definition of "function" to the definition of "rule", which is not really a well-defined concept.

Computer scientists have their own answer to this question: They don't define functions at all. Instead, they consider "function" to be one of those fundamental objects that are pre-existing. We will take another approach.

Before we define functions, we first define **relations**, which are more general.

1.2. Relations

Relations (to be specific: binary relations) are another concept that you are already familiar with in some form. Here are some examples:

- The relation $\subseteq$ is a relation between two sets. For example, $\{1,3\} \subseteq \{1,2,3,4\}$ but $\{1,5\} \not\subseteq \{1,2,3,4\}$.
- The order relations $\leq$ and $<$ and $\geq$ and $>$ are relations between two integers (or rational numbers, or real numbers). For example, $1 \leq 5$ but $1 \not\leq -1$.
- The containment relation $\in$ is a relation between an object and a set. For example, $5 \in \{1,2,3,4,5\}$ but $6 \notin \{1,2,3,4,5\}$.
- The divisibility relation $\mid$ is a relation between two integers.
- The relation "coprime" (aka "relatively prime") is a relation between two integers.
- In plane geometry, there are lots of relations: "parallel" (between two lines), "congruent" (between two shapes), "similar" (between two shapes), "directly similar", etc.
- For any given integer m, the relation "congruent modulo m" is a relation between two integers. Let me call it $\overset{m}{\equiv}$. Thus, $a \overset{m}{\equiv} b$ if and only if $a \equiv b \mod m$. For example, $2 \overset{2}{\equiv} 6$ but $2 \overset{2}{\not\equiv} 5$.

What do these relations all have in common? They can be applied to pairs of objects. Applying a relation to a pair of objects gives a statement that can be true or false.

A general relation R relates elements of a set X with elements of a set Y. To describe it, we need to know which pairs $(x, y) \in X \times Y$ do satisfy $x \mathrel{R} y$ and which pairs don't. Of course, it suffices to know which pairs do. In other words, we need to know the **set** of all pairs $(x, y) \in X \times Y$ that satisfy $x \mathrel{R} y$. So we can just take the relation R to **be** this set of pairs. This we will use as a definition of a relation:

Definition 1.2.1. Let X and Y be two sets. A **relation** between X and Y is a subset of $X \times Y$.

If R is a relation between X and Y, and $(x,y) \in X \times Y$, then

- we write $x \ R \ y$ if $(x,y) \in R$;
- we write $x \ \not R \ y$ if $(x,y) \notin R$.

All the familiar relations we have seen above can now be recast in terms of this definition (at least if they are defined on actual sets):

- The divisibility relation $\mid$ is a subset of $\mathbb{Z} \times \mathbb{Z}$, namely the subset
$$\begin{aligned}&\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \ \mid \ x \text{ divides } y\}\\ &= \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \ \mid \ \text{there exists some } z \in \mathbb{Z} \text{ such that } y = xz\}\\ &= \{(x,xz) \ \mid \ x \in \mathbb{Z} \text{ and } z \in \mathbb{Z}\}.\end{aligned}$$
- The coprimality relation ("coprime to") is a subset of $\mathbb{Z} \times \mathbb{Z}$, namely the subset
$$\begin{aligned}&\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \ \mid \ x \text{ is coprime to } y\}\\ &= \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \ \mid \ \gcd(x,y) = 1\}.\end{aligned}$$
- For any $m \in \mathbb{Z}$, the "congruent modulo m" relation $\overset{m}{\equiv}$ is a subset of $\mathbb{Z} \times \mathbb{Z}$, namely the subset
$$\begin{aligned}&\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \ \mid \ x \equiv y \bmod m\}\\ &= \{(x,x+mz) \ \mid \ x \in \mathbb{Z} \text{ and } z \in \mathbb{Z}\}.\end{aligned}$$
- Let P be the set of all points in the plane, and L be the set of all lines in the plane. Then, the "lies on" relation (as in "a point lies on a line") is a subset of $P \times L$, namely the subset
$$\{(p,\ell) \in P \times L \ \mid \ \text{the point } p \text{ lies on the line } \ell\}.$$
- The relation "adjacent to" as a relation between integers is a subset of $\mathbb{Z} \times \mathbb{Z}$, namely the subset
$$\begin{aligned}&\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \ \mid \ |x-y| = 1\}\\ &= \{(x,x+1) \ \mid \ x \in \mathbb{Z}\} \cup \{(x,x-1) \ \mid \ x \in \mathbb{Z}\}.\end{aligned}$$
- If A is any set, then the **equality relation** on A is the subset E_A of $A \times A$ given by
$$E_A = \{(x,y) \in A \times A \ \mid \ x = y\} = \{(x,x) \ \mid \ x \in A\}.$$
Two elements x and y of A satisfy $x \ E_A \ y$ if and only if they are equal.

- We can literally take any subset of $X \times Y$ and it will be a relation between X and Y. Just as with functions, a relation does not have to follow any "meaningful" rule. For example, here is a relation between $\{1,2,3\}$ and $\{5,6,7\}$:

  $$\{(1,6)\,,\ (1,7)\,,\ (3,5)\}\,.$$

  This relation can also be described by the following table:

	5	6	7
1	no	yes	yes
2	no	no	no
3	yes	no	no

  .

  So, if we call this relation R, then we have $1\ R\ 6$ and $1\ R\ 7$ and $3\ R\ 5$, but not (for example) $2\ R\ 5$.

1.3. Defining functions

We are now ready to give the rigorous definition of a function:

> **Definition 1.3.1.** Let X and Y be two sets. A **function** from X to Y means a relation R between X and Y that has the following property: For each $x \in X$, there exists **exactly one** $y \in Y$ such that $x\ R\ y$.

The above example

$$\{(1,6)\,,\ (1,7)\,,\ (3,5)\}$$

is therefore not a function, for two reasons:

- For $x = 1$, there exist **two** y's such that $x\ R\ y$ (namely, 6 and 7).
- For $x = 2$, there exists **no** y such that $x\ R\ y$.

Each of these reasons would be sufficient to disqualify this relation as a function. In the above list, only the equality relation is a function.