Math 220 Fall 2021, Lecture 17: Mathematical induction

1. Mathematical induction (cont'd)

1.1. Binomial coefficients

We shall now define one of the most important families of numbers in mathematics.

Recall that $n! = 1 \cdot 2 \cdot \cdots \cdot n$ for each $n \in \mathbb{N}$. In particular, $0! = 1$.

Definition 1.1.1. Let n and k be any numbers. Then, we define a number $\dbinom{n}{k}$ as follows:

- If $k \in \mathbb{N}$, then we set
$$\binom{n}{k} := \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k!}$$
(where the numerator is the product of k consecutive integers, the largest of which is n; you can also write it as $\prod\limits_{i=0}^{k-1}(n-i)$).

- If $k \notin \mathbb{N}$, then we set
$$\binom{n}{k} := 0.$$

This number $\dbinom{n}{k}$ is called "n **choose** k" and known as the **binomial coefficient** of n and k. Do not mistake the notation $\dbinom{n}{k}$ for a vector $\begin{pmatrix} n \\ k \end{pmatrix}$.

Example 1.1.2. For any number n, we have

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} = \frac{n(n-1)(n-2)}{6};$$
$$\binom{n}{2} = \frac{n(n-1)}{2!} = \frac{n(n-1)}{2};$$
$$\binom{n}{1} = \frac{n}{1!} = n;$$
$$\binom{n}{0} = \frac{(\text{empty product})}{0!} = \frac{1}{1} = 1;$$
$$\binom{n}{2.5} = 0;$$
$$\binom{n}{-1} = 0.$$

For any $k \in \mathbb{N}$, we have

$$\binom{0}{k} = \frac{0(0-1)(0-2)\cdots(0-k+1)}{k!} = \begin{cases} 0, & \text{if } k \neq 0; \\ 1, & \text{if } k = 0; \end{cases}$$
$$\binom{-1}{k} = \frac{(-1)(-2)(-3)\cdots(-k)}{k!} = (-1)^k \underbrace{\frac{1 \cdot 2 \cdot \cdots \cdot k}{k!}}_{=1} = (-1)^k.$$

Let us tabulate the values of $\binom{n}{k}$ for nonnegative integers n and k:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	1	0	0	0	0	0	0
$n=1$	1	1	0	0	0	0	0
$n=2$	1	2	1	0	0	0	0
$n=3$	1	3	3	1	0	0	0
$n=4$	1	4	6	4	1	0	0
$n=5$	1	5	10	10	5	1	0
$n=6$	1	6	15	20	15	6	1

.

What patterns can we spot in this table? (We are ignoring negative and non-integer n's for now.)

Proposition 1.1.3. Let $n \in \mathbb{N}$ and $k > n$. Then, $\dbinom{n}{k} = 0$.

Proof. If $k \notin \mathbb{N}$, this is clear by definition. Otherwise, again by definition, we have

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{0}{k!}$$

(since the product $n(n-1)(n-2)\cdots(n-k+1)$ has a factor of $n - n = 0$, and thus is 0). For example, for $n = 3$ and $k = 6$, we have

$$\binom{3}{6} = \frac{3 \cdot 2 \cdot 1 \cdot 0 \cdot (-1) \cdot (-2)}{6!} = \frac{0}{6!} = 0.$$

□

Because of the above proposition, our table has only 0s above the main diagonal, so we can redraw it as a triangular table (and fill in a few more rows while at that):

	$k=0$ ↙	$k=1$ ↙	$k=2$ ↙	$k=3$ ↙	$k=4$ ↙	$k=5$ ↙	$k=6$ ↙	$k=7$ ↙	
$n=0$ →	1								
$n=1$ →	1	1							
$n=2$ →	1	2	1						
$n=3$ →	1	3	3	1					
$n=4$ →	1	4	6	4	1				
$n=5$ →	1	5	10	10	5	1			
$n=6$ →	1	6	15	20	15	6	1		
$n=7$ →	1	7	21	35	35	21	7	1	
$n=8$ →	1	8	28	56	70	56	28	8	1

Theorem 1.1.4 (Pascal's identity, aka the recurrence of the binomial coefficients)**.** For any numbers n and k, we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. We are in one of the following three cases:

Case 1: The number k is a positive integer.

Case 2: The number k is 0.

Case 3: Neither of the above.

Consider Case 1 (this is the interesting case). Here, k is a positive integer, so that both k and $k-1$ belong to $\mathbb{N}$. Thus, the definition of binomial coefficients yields

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!};$$
$$\binom{n-1}{k-1} = \frac{(n-1)(n-2)(n-3)\cdots(n-k+1)}{(k-1)!};$$
$$\binom{n-1}{k} = \frac{(n-1)(n-2)(n-3)\cdots(n-k)}{k!}.$$

Therefore,

$$\begin{aligned}
&\binom{n-1}{k-1} + \binom{n-1}{k} \\
&= \frac{(n-1)(n-2)(n-3)\cdots(n-k+1)}{(k-1)!} + \frac{(n-1)(n-2)(n-3)\cdots(n-k)}{k!} \\
&= \frac{(n-1)(n-2)(n-3)\cdots(n-k+1)}{(k-1)!} + \frac{(n-1)(n-2)(n-3)\cdots(n-k)}{(k-1)!\cdot k} \\
&= \frac{(n-1)(n-2)(n-3)\cdots(n-k+1)\cdot k + (n-1)(n-2)(n-3)\cdots(n-k)}{(k-1)!\cdot k} \\
&= \frac{(n-1)(n-2)(n-3)\cdots(n-k+1)\cdot k + (n-1)(n-2)(n-3)\cdots(n-k)}{k!} \\
&= \frac{(n-1)(n-2)(n-3)\cdots(n-k+1)\cdot(k+(n-k))}{k!} \\
&= \frac{(n-1)(n-2)(n-3)\cdots(n-k+1)\cdot n}{k!} \\
&= \frac{n(n-1)(n-2)(n-3)\cdots(n-k+1)}{k!} = \binom{n}{k},
\end{aligned}$$

so the theorem is proved in Case 1.

In Case 2, we have $k=0$. The equation we want to prove

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

therefore simplifies to

$$\binom{n}{0} = \binom{n-1}{-1} + \binom{n-1}{0}.$$

But this is just saying that $1 = 0 + 1$, which is true.

Case 3 is even more trivial: In this case, neither k nor $k-1$ belongs to $\mathbb{N}$, so that our equation boils down to $0 = 0 + 0$. $\square$

Theorem 1.1.5 (factorial formula)**.** Let $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. Then,

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

Proof. The definition of $\binom{n}{k}$ yields

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

However,

$$\begin{aligned} n(n-1)(n-2)\cdots(n-k+1) &= \frac{n(n-1)(n-2)\cdots 1}{(n-k)(n-k-1)\cdots 1} \\ &= \frac{1 \cdot 2 \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot (n-k)} = \frac{n!}{(n-k)!}. \end{aligned}$$

Substituting this into the previous equation, we obtain

$$\binom{n}{k} = \frac{\left(\dfrac{n!}{(n-k)!}\right)}{k!} = \frac{n!}{k! \cdot (n-k)!}.$$

□

This factorial formula is the most direct way to express $\binom{n}{k}$ when $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. However, it cannot be used to compute $\binom{-1}{k}$ or $\binom{1/2}{k}$.

Theorem 1.1.6 (symmetry of Pascal's triangle)**.** Let $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. Then,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof. The factorial formula yields

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} \qquad \text{and}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)! \cdot (n-(n-k))!} = \frac{n!}{(n-k)! \cdot k!} = \frac{n!}{k! \cdot (n-k)!}.$$

The right hand sides are equal, so the left hand sides are also equal. □

Warning: This symmetry does **not** hold for negative (or non-integer) n's. For example, $\binom{-1}{k} \neq \binom{-1}{-1-k}$ usually.

Corollary 1.1.7. For any $n \in \mathbb{N}$, we have $\binom{n}{n} = 1$.

Proof. Applying the previous theorem to $k = n$, we obtain $\binom{n}{n} = \binom{n}{n-n} = \binom{n}{0} = 1$. □

Note that the corollary again does **not** hold for negative (or non-integer) n's. For example, $\binom{-1}{-1} = 0 \neq 1$.

The perhaps simplest pattern in Pascal's triangle is that all entries are integers! This is not obvious from the original definition of the binomial coefficients, but we can prove it now:

Theorem 1.1.8. For any $n \in \mathbb{N}$ and any number k, we have $\binom{n}{k} \in \mathbb{N}$.

Proof. We induct on n:

Base case: The theorem holds for $n = 0$, since

$$\binom{0}{k} = \begin{cases} 0, & \text{if } k \neq 0; \\ 1, & \text{if } k = 0 \end{cases} \in \mathbb{N}.$$

Induction step: Let n be a positive integer. Assume (as the induction hypothesis) that $\binom{n-1}{k} \in \mathbb{N}$ for all numbers k. We must prove that $\binom{n}{k} \in \mathbb{N}$ for all numbers k. (This is a "step from $n-1$ to n".)

This can be achieved immediately using Pascal's identity:

$$\binom{n}{k} = \underbrace{\binom{n-1}{k-1}}_{\substack{\in \mathbb{N} \\ \text{(by the induction hypothesis)}}} + \underbrace{\binom{n-1}{k}}_{\substack{\in \mathbb{N} \\ \text{(by the induction hypothesis)}}} \in \mathbb{N}.$$

This completes the induction step. Thus, the theorem is proved. □

The last theorem might look rather surprising the way we proved it. There is another proof that explains it a lot better. Namely, we can interpret $\binom{n}{k}$ as a number of certain things:

Theorem 1.1.9 (combinatorial interpretation of binomial coefficients)**.** Let $n \in \mathbb{N}$ and k be a number. Let A be any n-element set. (Here, an "n-element set" means a set that has exactly n distinct elements. For example, $\{2, 6, 11\}$ is a 3-element set, and this doesn't change if I rewrite this set as $\{2, 6, 2, 11\}$.)

Then,

$$\binom{n}{k} \text{ is the number of all } k\text{-element subsets of } A.$$

Example 1.1.10. Let $n = 4$ and $k = 2$ and $A = \{1, 2, 3, 4\}$. Then, the 2-element subsets of A are

$$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\},$$

so there are 6 of them. And indeed, $\binom{n}{k} = \binom{4}{2} = 6$.

For another example, the 3-element subsets of $\{1, 2, 3, 4, 5\}$ are

$$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\},$$
$$\{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}.$$

There are 10 of them. And indeed, $\binom{5}{3} = 10$.

Theorem 1.1.9 will be proved later in this course[1], as we learn more about how to deal with finite sets and what it formally means for a set to have k elements. Note that the k-element subsets of A are also known as **combinations without replacement**.

Note also that Theorem 1.1.9 is the reason why $\binom{n}{k}$ is called "n choose k". The word "choose" refers to choosing k distinct elements from the n-element set A.

[1]**Note (edit):** This did not end up happening in Fall 2021, but should be part of the course in a normal quarter.