# Math 220 Fall 2021, Lecture 12: Mathematical induction

## 1. Mathematical induction (cont'd)

### 1.1. What is induction? (cont'd)

Last time, we stated the following principle (our last deduction rule):

> **Principle of Mathematical Induction** (or, short, **Principle of Induction):**
>
> Let $P(n)$ be a predicate that depends on a variable $n$, which is supposed to be a nonnegative integer.
>
> Assume that you have proved $P(0)$.
>
> Assume further that you have proved[1]
>
> $$\forall n \in \mathbb{N} : (P(n) \Longrightarrow P(n+1)).$$
>
> Then, you can deduce that
>
> $$\forall n \in \mathbb{N} : P(n).$$

Intuitively, this should be fairly plausible. For example, you can deduce that $P(4)$ holds, because you have $P(0)$ and $P(0) \Longrightarrow P(1)$ and $P(1) \Longrightarrow P(2)$ and $P(2) \Longrightarrow P(3)$ and $P(3) \Longrightarrow P(4)$, so you can go from $P(0)$ to $P(4)$.

Let us see an example of a proof using the principle of mathematical induction. We shall prove the following theorem:

**Theorem 1.1.1.** Let $n \in \mathbb{N}$. Let $s_n$ be the sum of the first $n$ positive integers. In other words, let

$$s_n := 1 + 2 + 3 + \cdots + n.$$

Then,

$$s_n = \frac{n(n+1)}{2}.$$

**Example 1.1.2.** For example, for $n = 5$, we have $s_n = s_5 = 1 + 2 + 3 + 4 + 5$, and the theorem claims that $s_5 = \frac{5 \cdot 6}{2}$. And indeed, you can easily check that both sides are equal to 15.

[1] Recall that $\mathbb{N}$ means the set of all nonnegative integers, i.e., the set $\{0, 1, 2, \ldots\}$.

For another example, for $n = 2$, we have $s_n = s_2 = 1 + 2$, and the theorem claims that $s_2 = \frac{2 \cdot 3}{2}$.

For another example, for $n = 1$, we have $s_n = s_1 = 1$, and the theorem claims that $s_1 = \frac{1 \cdot 2}{2}$.

For another example, for $n = 0$, we have $s_n = s_0 = 0$, since it is agreed that an empty sum (i.e., a sum with no addends) is 0 by definition. And the theorem agrees again, since it claims that $s_0 = \frac{0 \cdot 1}{2}$.

*Proof of the Theorem.* We denote the predicate $s_n = \frac{n(n+1)}{2}$ by $P(n)$. So we want to show that

$$\forall n \in \mathbb{N} : P(n).$$

By the Principle of Mathematical Induction, we will achieve this once we have

- proved $P(0)$;
- proved $\forall n \in \mathbb{N} : (P(n) \Longrightarrow P(n+1))$.

Proving $P(0)$ is easy: Indeed, $P(0)$ is saying that $s_0 = \frac{0(0+1)}{2}$, and this can be checked directly by computing both sides ($s_0 = 0$ and $\frac{0(0+1)}{2} = 0$).

Now it remains to prove $\forall n \in \mathbb{N} : (P(n) \Longrightarrow P(n+1))$.

To do so, we let $n \in \mathbb{N}$ be arbitrary. We must prove $P(n) \Longrightarrow P(n+1)$. So we assume that $P(n)$ holds. We must then prove that $P(n+1)$ holds.

We have assumed that $P(n)$ holds. In other words, $s_n = \frac{n(n+1)}{2}$.

We need to prove that $P(n+1)$ holds. In other words, we need to prove that $s_{n+1} = \frac{(n+1)((n+1)+1)}{2}$.

To do this, we compare

$$\begin{aligned} s_{n+1} &= 1 + 2 + \cdots + n + (n+1) \qquad \text{with} \\ s_n &= 1 + 2 + \cdots + n. \end{aligned}$$

The first sum contains all addends of the second sum, but also a new addend, which is $n+1$. So

$$\begin{aligned} s_{n+1} &= s_n + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \qquad \left(\text{since we assumed that } s_n = \frac{n(n+1)}{2}\right) \\ &= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}. \end{aligned}$$

Thus, we have proved that $P(n+1)$ holds.

So we have proved the implication $P(n) \Longrightarrow P(n+1)$. Since we have proved it for all $n \in \mathbb{N}$, we thus conclude that $\forall n \in \mathbb{N} : (P(n) \Longrightarrow P(n+1))$. This is precisely what was missing. So now we can apply the Principle of Mathematical Induction, and conclude that $\forall n \in \mathbb{N} : P(n)$. The theorem is thus proved. $\square$

The above was an example of a **proof by mathematical induction** (or, for short, a **proof by induction**, or just an **induction proof**). Such proofs are very frequent in math. (Note that this is not what philosophers call "induction".)

Here are some standard pieces of terminology that are commonly used in proofs by induction. Let's say you are proving a statement of the form $\forall n \in \mathbb{N} : P(n)$.

- The $n$ is called the **induction variable**, and you say that you **induct on** $n$. Actually, the induction variable doesn't have to be called $n$. You can just as well call it $k$ or $x$ or $\alpha$ or ✠ or Æ.
- The proof of $P(0)$ is called the **induction base** or the **base case**. In our above proof, the base case was showing that $s_0 = \dfrac{0(0+1)}{2}$.
- The proof of $\forall n \in \mathbb{N} : P(n) \Longrightarrow P(n+1)$ is called the **induction step**. In our above proof, the induction step was when we assumed that $s_n = \dfrac{n(n+1)}{2}$ and proved that $s_{n+1} = \dfrac{(n+1)((n+1)+1)}{2}$. In the induction step, the assumption $P(n)$ is called the **induction assumption** or the **induction hypothesis**, and the claim $P(n+1)$ (that you are trying to prove) is called the **induction goal**.

Let us rewrite our above proof using this language:

*Proof of the Theorem.* We induct on $n$.

*Base case:* The theorem holds for $n = 0$, because $s_0 = \dfrac{0(0+1)}{2}$ can be checked directly by computing both sides ($s_0 = 0$ and $\dfrac{0(0+1)}{2} = 0$).

*Induction step:* Let $n \in \mathbb{N}$. We assume that the theorem holds for $n$ (this is what we previously called $P(n)$). We will now show that it also holds for $n+1$ (this is what we previously called $P(n+1)$).

We have assumed that the theorem holds for $n$. In other words, $s_n = \dfrac{n(n+1)}{2}$.

We need to prove that the theorem holds for $n+1$. In other words, we need to prove that $s_{n+1} = \dfrac{(n+1)((n+1)+1)}{2}$.

To do this, we compare

$$\begin{aligned} s_{n+1} &= 1+2+\cdots+n+(n+1) \qquad \text{with} \\ s_n &= 1+2+\cdots+n. \end{aligned}$$

The first sum contains all addends of the second sum, but also a new addend, which is $n+1$. So

$$\begin{aligned} s_{n+1} &= s_n + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \qquad \left(\text{since we assumed that } s_n = \frac{n(n+1)}{2}\right) \\ &= \frac{n(n+1)+2(n+1)}{2} = \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}. \end{aligned}$$

Thus, we have proved that the theorem holds for $n+1$. This completes the induction step, and thus the theorem is proved. □

What does "$1+2+\cdots+n$" actually mean? Actually, $s_n = 1+2+\cdots+n$ is defined **recursively**. That is, to define $s_n$ for a given $n$, we take the previous value $s_{n-1}$ and we add $n$ to it (unless $n=0$, in which case there is nothing to sum and we just set $s_0 := 0$). So the very definition of "$1+2+\cdots+n$" has the same structure as our above proof by induction: Instead of giving you the result right away, it tells you how to obtain each value from the preceding one. This is called a **recursive definition**.

Let us see another recursive definition. We will define a famous sequence of integers known as the **Fibonacci sequence**:

**Definition 1.1.3.** The **Fibonacci sequence** is the sequence $(f_0, f_1, f_2, \ldots)$ of nonnegative integers defined recursively by setting

$$\begin{aligned} &f_0 := 0, \qquad f_1 := 1, \qquad \text{and} \\ &f_n := f_{n-1} + f_{n-2} \qquad \text{for each } n \geq 2. \end{aligned}$$

Once again, this definition is recursive, so it doesn't let you compute $f_n$ immediately in one step, but rather tells you how to compute $f_n$ if the previous entries of this sequence have already been computed. But this still allows you to compute $f_n$; you just need to compute all previous entries first. Let us do this for the first 10 or so entries:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 |

.

As we see, a recursive definition is a perfectly fine way of defining (e.g.) a sequence of numbers. It lets you compute each entry of the sequence. Note that it is important that $f_n$ is defined in terms of the previous entries – not in terms of $f_n$ itself or of the next entries; that would be circular. So, for example, if we set

$$f_n := f_{n+1} - f_{n-2} \qquad \text{instead of } f_n := f_{n-1} - f_{n-2},$$

then we could not even compute $f_2$, because we would need more and more uncomputed values.

Let us see some properties of the Fibonacci numbers (i.e., are the entries of the Fibonacci sequence).

**Theorem 1.1.4.** For any $n \in \mathbb{N}$, we have

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

**Example 1.1.5.**

$$\begin{aligned} 1+1+2+3+5+8 &= 20 = 21 - 1; \\ 1+1+2+3+5+8+13 &= 33 = 34 - 1; \\ 1+1+2+3+5+8+13+21 &= 54 = 55 - 1. \end{aligned}$$

*Proof of the Theorem.* We induct on $n$:

*Base case:* The theorem is true for $n = 0$, because it is saying that $0 = f_2 - 1$.

*Induction step:* Let $n \in \mathbb{N}$.

Assume that the theorem is true for $n$. In other words, we assume that

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

We must prove that the theorem is also true for $n + 1$. In other words, we must prove that

$$f_1 + f_2 + \cdots + f_{n+1} = f_{(n+1)+2} - 1.$$

To do so, we just compute

$$\begin{aligned} f_1 + f_2 + \cdots + f_{n+1} &= \underbrace{(f_1 + f_2 + \cdots + f_n)}_{= f_{n+2} - 1} + f_{n+1} = (f_{n+2} - 1) + f_{n+1} \\ &= \underbrace{(f_{n+2} + f_{n+1})}_{\substack{= f_{n+3} \\ \text{(by the recursive definition} \\ \text{of the Fibonacci sequence)}}} - 1 = f_{n+3} - 1. \end{aligned}$$

So the induction step is complete. □

**Theorem 1.1.6.** We have

$$f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1} \qquad \text{for any } n, m \in \mathbb{N}.$$

*Proof.* Next time. Idea: Induct on $n$ (or on $m$ if you wish).

*Base case:* We must show that the theorem holds for $n = 0$. In other words, we must show that

$$f_{0+m+1} = f_0 f_m + f_{0+1} f_{m+1} \qquad \text{for any } m \in \mathbb{N}.$$

But this follows by comparing

$$f_{0+m+1} = f_{m+1} \qquad \text{with}$$
$$\underbrace{f_0}_{=0} f_m + \underbrace{f_{0+1}}_{=f_1=1} f_{m+1} = 0 f_m + 1 f_{m+1} = f_{m+1}.$$

*Induction step:* Next time. $\square$